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2.1

Defn An $m \times n$ matrix is an array of mn entries written in m rows & n columns.

Ex

$$\begin{array}{cc} \text{2 rows} & \underbrace{\begin{bmatrix} 2 & -1 & 6 \\ 1 & 4 & -3 \end{bmatrix}}_{\text{3 columns}} \end{array}$$

2×3 matrix.

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$$

column vectors

$$\vec{v}_i = \left.\begin{bmatrix} \vdots \\ \vdots \end{bmatrix}\right\} m \text{ entries}$$

$\vec{v}_i \in \mathbb{R}^m$

$i = 1, 2, 3, \ldots n.$

$$A = \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}$$

i th row

j th column

a_{ij}, or A_{ij}

Ex

$$A = \begin{bmatrix} 3 & -1 & 5 \\ 6 & 0 & 11 \end{bmatrix}$$

$A_{12} = -1$

$A_{21} = 6$

$$\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{kk} \end{bmatrix} \rightarrow \text{main diagonal}$$

$$\begin{bmatrix} 2 & 11 & 7 \\ 0 & -1 & 32 \\ 1 & 7 & 6 \end{bmatrix}$$

Diagonal matrix : • Square matrix, and
• All entries off the main diagonal are 0:

$$A_{ij} = 0 \quad \text{if} \quad i \neq j$$

Ex

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

ok

n×n matrix

• Identity matrix I_n → main diagonal all 1's.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ off diagonal all zeros.

Scalar multiples of matrices

$$(-2)\begin{bmatrix} 5 & 1 & 0 \\ 6 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -10 & -2 & 0 \\ -12 & 2 & -6 \end{bmatrix}$$

Defn. • $(rA)_{ij} = r A_{ij}$ for all matrices A

• $(A+B)_{ij} = A_{ij} + B_{ij}$

provided that both A and B are both $m \times n$, i.e. same dimensions.

When we are adding two matrices of the same dimensions, say A and B, The i^{th} row j^{th} column of $A+B$ is obtained by adding i^{th} row, j^{th} column entry of A to i^{th} row, j^{th} column entry of B.

$$\begin{bmatrix} 2 & 1 \\ -1 & 6 \\ 11 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 7 & 0 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2+(-1) & 1+6 \\ -1+7 & 6+0 \\ 11+5 & 4+(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 6 & 6 \\ 16 & 2 \end{bmatrix}$$

Ex

$$(-2)\underbrace{\begin{bmatrix} 1 & 5 & 1 \\ 3 & 4 & 0 \\ -1 & 6 & 4 \end{bmatrix}}_{3\times 3} + 7\underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 6 \\ 0 & 1 & 5 \end{bmatrix}}_{3\times 3}$$

You can do directly

OR LONG WAY

$$= \begin{bmatrix} 12 & 11 & -9 \\ 1 & -8 & 42 \\ 2 & -5 & 27 \end{bmatrix} \longleftarrow \begin{bmatrix} -2 & -10 & -2 \\ -6 & -8 & 0 \\ 2 & -12 & -8 \end{bmatrix} + \begin{bmatrix} 14 & 21 & -7 \\ 7 & 0 & 42 \\ 0 & 7 & 35 \end{bmatrix}$$

Matrix Multiplication:

Recall $$A \cdot \vec{c} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

column vectors

real numbers

$$= c_1 \vec{a}_1 + c_2 \vec{a}_2 + \cdots + c_n \vec{a}_n$$

Ex:

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1)\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -3 \\ +1 \end{bmatrix} + \begin{bmatrix} -2 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ +5 \\ 5 \end{bmatrix}$$

Def Let A be an $m \times n$ matrix & B be an $p \times q$ matrix.

If $n \neq p$, then AB is not defined.

If $n = p$, then AB is a $m \times q$ matrix:

$$m \times \underbrace{n \quad p}_{=} \times q$$

$$AB = A \cdot \left[\vec{b}_1 \; \vec{b}_2 \; \cdots \; \vec{b}_q \right]$$ columns of B

$$= \left[A \cdot \vec{b}_1 \quad A \cdot \vec{b}_2 \quad A \cdot \vec{b}_3 \quad \quad A\vec{b}_q \right]$$

each calculate as above

each is a column of AB

Calculate using the definition on p 5.

Ex

$$\begin{bmatrix} -2 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ -1 & 1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 29 & 54 \end{bmatrix}.$$

$2 \times 3 \quad 3 \times 2 \quad \checkmark \qquad 2 \times 2$

$$\begin{bmatrix} -2 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 13 \\ 29 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 54 \end{bmatrix}$$

<u>Row – Column Rule</u> (More effective, shorter to write)

- Let A be an $m \times n$ matrix
- Let B be a $p \times q$ matrix.
- $\times$ $p = n$

Then AB is a $m \times q$ matrix whose i^{th} row j^{th} column entry $(A \cdot B)_{ij}$

is the product of the i^{th} row of A with j^{th} column of B

which is obtained by the sum of the products of the corresponding entries as follows:

$$\underbrace{(A\cdot B)_{ij}}_{\substack{i^{th}\text{ row}\\ j^{th}\text{ column}\\ \text{entry of } AB}} = \underbrace{\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix}}_{i^{th}\text{ row of } A} \underbrace{\begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix}}_{j^{th}\text{ column of } B}$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

Ex

$$A\cdot B = \underbrace{\overbrace{\begin{bmatrix} 1 & 0 \\ -1 & 6 \\ 4 & 2 \end{bmatrix}}^{A}}_{3\times 2} \underbrace{\overbrace{\begin{bmatrix} -2 & 1 & 4 \\ 5 & 0 & -2 \end{bmatrix}}^{B}}_{2\times 3} = AB = \underbrace{\begin{bmatrix} -2 & 1 & 4 \\ 32 & -1 & -16 \\ 2 & 4 & 12 \end{bmatrix}}_{3\times 3}$$

A_{11} $\quad A_{12}$ $\quad A_{21}$

✓

For A_{11}

$$\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\text{row 1 of } A} \underbrace{\begin{bmatrix} -2 \\ 5 \end{bmatrix}}_{\text{column 1 of } B} = (1\cdot -2) + (0\cdot 5) = -2 = A_{11}$$

$$\underbrace{\begin{bmatrix} -1 & 6 \end{bmatrix}}_{\text{row 2 of } A} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = (-1\cdot -2) + (6\cdot 5) = 32 = A_{21}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\text{row 1 of } A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{column 2 of } B} = 1\cdot 1 + 0\cdot 0 = 1 = A_{12}$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}}_{B} = \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix} = AB$$

$$\nparallel$$

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 11 & 7 \end{bmatrix} = BA$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 6 & 0 & 3 \end{bmatrix}}_{2\times 3} \quad \text{B not defined}$$

$$3 \neq 2$$

Transpose writing rows as columns

Formal: $(A^T)_{ij} = A_{ji}$

Ex

$$\begin{bmatrix} 1 & 4 \\ 2 & -7 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -7 & 6 \end{bmatrix}$$

Ex

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix} 1 & 1 & 4 \\ 6 & 0 & 3 \end{bmatrix}^T}_{\substack{(2\times 3)^T \\ 3\times 2}}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}}_{2\times 3}\underbrace{\begin{bmatrix} 1 & 6 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 10 & 12 \\ 21 & 15 \end{bmatrix}}_{2\times 2}$$