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1.7 Linear Independence

Defn An indexed set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}$ in $\mathbb{R}^n$ is called linearly independent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0} \text{ is true}$$

ONLY for $c_1 = c_2 = c_3 = \cdots = c_p = 0$.

real
$c_i \in \mathbb{R}$.
$\vec{v}_i \in \mathbb{R}^n$
vectors

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_p\}$ is called dependent if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = 0 \text{ is true}$$

for a solution $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Remarks

1. $$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2. indexed means: ordered and may repeat.

In an unindexed set: $\{1, 0, 1, 3, 2\} = \{2, 1, 3, 0\}$

not ordered; repetition of an element doesn't bring in a new element

Easy Examples:

① Is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$ linearly independent or dependent ?

Sol'n: $\begin{bmatrix} 2 \\ 5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\underbrace{2}_{c_1} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{v}_1} + \underbrace{3}_{c_2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}_2} + \underbrace{(-1)}_{c_3} \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{v_3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

ANSWER: Dependent

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0} \qquad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

& not all c_1, c_2, c_3 are 0

TYPO: (at least one of c_1, c_2, c_3 is NOT 0)

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opposites

$\underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$	$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$
For All k, $k = 1, 2, 3, \ldots, p$,	there exist at least one k s.t.
$c_k = 0$	$c_k \neq 0$

2 Is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ linearly dependent or independent?

$$\underbrace{c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

possible only when $c_1 = c_2 = 0$, there is no other solution. INDEPENDENT!

Ex 3 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$ Q: Is it linearly independent?

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

I know $c_1 = c_2 = c_3 = 0$ is a solution.
not sufficient

Relevant question: Is there another solution?

How? Row Reduce! PTO

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right] \longrightarrow \longrightarrow \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

free parameter

$$c_1 = -3c_3$$
$$c_2 = 2c_3$$
$$c_3 = \text{free}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = c_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Sol}^{n} \text{ set} = \text{Span}\left(\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right)$$

Take any $c_3 \neq 0$, you will get a non-trivial sol^{n}.

<u>So:</u> $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$ is linearly <u>dependent</u>.

<u>Thm:</u> Columns of a matrix A form a linearly independent set

Book

$\Longleftrightarrow$ $A \cdot \vec{x} = 0$ has <u>only</u> the trivial sol^{n}.

not in the book

$\Longleftrightarrow$ RREF of A (no augmentation) has a leading 1 in every column.

Exc ~ #14

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix} \right\}$$

determine h for which these vectors are dependent / OR independent.

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 1 \\ -4 & 6 & h \end{bmatrix} \xrightarrow[\text{RR}]{} \xrightarrow[\text{~~HW~~ to check}]{} \begin{bmatrix} 1 & 0 & 17 \\ 0 & 1 & 5 \\ 0 & 0 & h+38 \end{bmatrix}$$

Case 1 $\quad h = -38 \quad$ 3^{rd} column has no leading 1

$\Longrightarrow$ S is linearly dependent

Case 2 $\quad h \neq -38$

$$\xrightarrow[\text{further}]{\text{RR}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ S is linearly independent

Some easy short-cuts

- Any set containing $\vec{0}$ is linearly <u>dependent</u>

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \quad \text{dependent.}$$

$$\underset{c_1 \neq 0}{5} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Any "indexed" set repeating a vector is linearly dependent.

$$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \quad \text{dependent}$$

$$1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

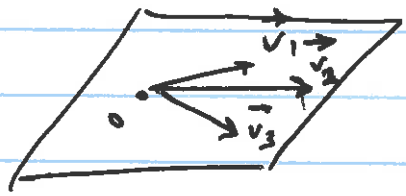
- $\{ \vec{v}_1, \vec{v}_2 \}$ linearly independent

$\iff$ <u>neither</u> $\vec{v}_1$ nor $\vec{v}_2$ is 0, and $\vec{v}_1$ & $\vec{v}_2$ are not scalar multiples of each other

<u>i.e.</u> not parallel

- $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent

  $\iff$ the tips of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{0}$ are coplanar



- In $\mathbb{R}^n$ any collection of $n+1$ or more vectors form a dependent set

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} \pi \\ e \\ 7 \end{bmatrix}, \begin{bmatrix} \cos 88 \\ e^{57} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

4 vectors in $\mathbb{R}^3$ are linearly <u>dependent</u>

see earlier Thm.

A is 3×4 matrix

$$\text{3 rows} \underbrace{\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}}_{\text{4 columns}}$$

You can have at most 3 leading 1's of a row

there is a column without a leading 1 of a row.

***** Review/Overview

Let $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ be an indexed set of vectors in $\mathbb{R}^n$.

Let $A = [\vec{v}_1 \; \vec{v}_2 \; \dots \; \vec{v}_p]$ be the $n \times p$ matrix whose columns are $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p$.

Row reduce A, get RREF(A).

1.5 (A) $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ span $\mathbb{R}^n$

$\iff$ Every row of RREF(A) has a leading 1.

1.7 (B) $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_p\}$ is linearly independent

$\iff$ Every column of RREF(A) has a leading 1 of a row

(A & B) $\{\vec{v}_1, \dots \vec{v}_p\}$ is a basis of $\mathbb{R}^n$: Def^n $\left\{\begin{array}{l}\text{linearly independent} \\ \text{and} \\ \text{spans } \mathbb{R}^n\end{array}\right.$

$\iff$ $\boxed{\text{Every row and column of RREF(A) has a leading 1 of a row}}$

$$\iff \left\{\begin{array}{l} \text{RREF}(A) = \underbrace{\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}}_{p} \Big\}\, n \quad \text{Identity matrix} \\ n = p. \end{array}\right.$$