Feb 4, 2020

①

1.4

Defn Let A be an $m \times n$ matrix
(an array of mn entries placed in m rows & n columns)
(usually $[\ \ ]$, $(\ \ )$ are used but never $|\ \ |$, $\{\ \ \}$)

. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$. Then

Define $$A \cdot \vec{x} = [\underbrace{\vec{a}_1}_{\text{column 1 of } A} \ \vec{a}_2 \cdots \underbrace{\vec{a}_n}_{\text{column } n \text{ of } A}] \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \left[\vec{a}_1\right] + x_2 \left[\vec{a}_2\right] + \cdots + x_n \left[\vec{a}_n\right]$$

Ex $$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+6-4 \\ 3+4+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

OR

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 3 \cdot 2 + (-4) \cdot 1 \\ 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Ex. b)

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 4 & 7 \\ 6 & 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + x_3 \\ -x_1 + 4x_2 + 7x_3 \\ 6x_1 + 4x_2 + 3x_3 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

Ex. c)

$$\begin{bmatrix} 2 & 3 & 1 & -4 \\ 7 & 6 & 1 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} ? \\ ? \end{bmatrix}$$? size ?

$$\underbrace{2 \times 4 \qquad 4}_{} \times 1 \qquad 2 \times 1$$

$$1\begin{bmatrix} 2 \\ 7 \end{bmatrix} + 2 \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 8 \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 7 \end{bmatrix} + \begin{bmatrix} 6 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -32 \\ 64 \end{bmatrix}$$

$$= \begin{bmatrix} -24 \\ 83 \end{bmatrix}$$

Thm If A is $m \times n$ matrix with columns $\underbrace{\vec{a}_1\ \vec{a}_2 \ldots \vec{a}_n}$:

$$m \text{ rows} \left\{ \underbrace{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}}_{n \text{ columns}} = A \, ; \ \vec{a}_i \in \mathbb{R}^m \right.,$$

Then

The matrix equation $A \cdot \vec{x} = \vec{b}$

- has the same solution of the equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b},$$ and

- which in turn has the same solution of the SLE whose augmented matrix is

$$\left[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n \,\middle|\, b \right].$$

PTO for example

Example

Solve $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$

and $b = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

Solve $x_1 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

Solve $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{array}\right]$

All are asking the same question in essence.

So Let's solve it:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ -3 & -4 & 2 & 2 \\ 5 & 2 & 3 & -3 \end{array}\right] \xrightarrow{3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 5 & 2 & 3 & -3 \end{array}\right]$$

$$\xrightarrow{R_3-5R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & -8 & 8 & -8 \end{array}\right] \xrightarrow{-\frac{1}{8}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 2 & -1 & 5 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \Leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 5 \end{array}\right] \xrightarrow{R_1-2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & 5 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1-3R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

what does this tell me?

PTO

The row reduction on the previous page tells me:

(1) $$\begin{bmatrix} 1 & 2 & -1 \\ -3 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ +2 \\ -3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 3 \end{bmatrix}$$

(2) $$-4 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ +2 \\ -3 \end{bmatrix}.$$

Defn Given $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p \in \mathbb{R}^n$

$\text{Span}(\vec{v}_1, \vec{v}_2, \dots \vec{v}_p)$

$$= \{ c_1\vec{v}_1 + \dots + c_p\vec{v}_p \mid c_1, c_2 \dots c_p \in \mathbb{R} \}$$

The set of all possible linear combinations of $\vec{v}_1 \dots \vec{v}_p$.

Ex. Is every $\vec{b} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ in the span of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$?

Reword it: Suppose A, B, C given, Do there exist x_1, x_2 s.t.

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \ ?$$

P③ Thm says

Solve SLE: $\left[\begin{array}{cc|c} 1 & 1 & A \\ 1 & 2 & B \\ 0 & 0 & C \end{array}\right] \xrightarrow{RR} ??$

$$\boxed{\begin{bmatrix} A \\ B \\ C \neq 0 \end{bmatrix} \notin \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)}$$

not in

$\Uparrow$

- If $C \neq 0$, this system has no sol'n.

Suppose $C = 0$, then what happens?

$$\left[\begin{array}{cc|c} 1 & 1 & A \\ 1 & 2 & B \\ 0 & 0 & C=0 \end{array}\right] \xrightarrow[RR]{} \left[\begin{array}{cc|c} 1 & 0 & 2A-B \\ 0 & 1 & B-A \\ 0 & 0 & 0 \end{array}\right]$$

If $C=0$ then I can write

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = (2A-B) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (B-A) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$C = 0$

$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \in$ span of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

Ex. 6 Is every $\vec{b} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ in the span of

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \text{ ? } \quad \underline{\text{YES}}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

*** **Thm**: Let A be an $m \times n$ matrix

For <u>all</u> $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.	$\Longleftrightarrow$	For <u>all</u> $\vec{b} \in \mathbb{R}^m$, b is a linear combination of the columns of A
$\Updownarrow$		$\Updownarrow$
RREF of A has a leading 1 (pivot) in every row.	$\Longleftrightarrow$	The columns of A span $\mathbb{R}^m$

PTO for an example.

Ex

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\cdots} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Not all steps are shown.

RREF of A has a leading 1 in every row

Conclusions from Thm p8

- $\text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right) = \mathbb{R}^3.$

- Every $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ can be written as equal to

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

for some c_1, c_2, c_3 (depending on A, B, C)

- SLE
$$\left.\begin{aligned} x_1 + 2x_3 &= A \\ 2x_1 + x_2 + x_3 &= B \\ x_1 + x_2 \quad &= C \end{aligned}\right\}$$
is consistent for every choice of A, B, C