22M:55

OLD FINALS

22M:55, Section A-Old FINAL EXAMS, 2006

1. a. Determine whether $\lim_{x\to 0}\left|\sin\frac{1}{x}\right|$ exists or not. Justify your answer by applying theorems or by providing a proof.

b. Prove that $\lim_{x\to 5}\left(x^2-3x+1\right)=11$ directly from the definition of the limit $(\forall\varepsilon\exists\delta...)$.

2. a. State the definition of *continuity* of a function $f: I\to\mathbf{R}$ at a point $c\in I$.

b. Prove the following (Theorem 21.12).

Let $f: I\to J$ and $g: J\to\mathbf{R}$, where I and J are intervals in $\mathbf{R}$. If f is continuous at $c\in I$ and g is continuous at $f(c)$, then $g\circ f: I\to\mathbf{R}$ is continuous at c.

3. Let $h(x)=\begin{cases} x^2\cdot\cos\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}:\mathbf{R}\to\mathbf{R}$.

a. Is the function $h(x)$ differentiable at $x=0$? Prove your assertion.

b. Prove the following (Theorem 25.7b).

Let $f: I\to\mathbf{R}$ and $g: I\to\mathbf{R}$, where I is an interval in $\mathbf{R}$. If both f and g are differentiable at $c\in I$, then $f+g: I\to\mathbf{R}$ is differentiable at c and $(f+g)'(c)=f'(c)+g'(c)$.

4. a. State *the Mean Value Theorem.*

b. You may assume that the derivative of $\sin x$ is $\cos x$, and both of them are bounded by 1 in absolute value. Prove that $\left|\frac{\sin am-\sin bm}{m}\right|\leq|a-b|$, if $m\neq 0$.

5. a. State *the Intermediate Value Theorem.*

b. Prove that the equation $e^x=-3x$ has at least one solution in $\mathbf{R}$.

c. Prove that the equation $e^x=-3x$ has a unique (at most one) solution in $\mathbf{R}$.

You may assume that the derivative of e^x is e^x, and $e^x>0$.

6. a. Prove the following (Theorem 22.10). If $f:[a,b]\to\mathbf{R}$ is continuous, then $f([a,b])=[m,M]$ for some $m,M\in\mathbf{R}$. Identify each theorem you use in this proof.

b. To avoid confusion of notation: $(a,b)=\{x\in\mathbf{R}: a<x<b\}$ means an open interval.

Given the statement "If $f:(a,b)\to\mathbf{R}$ is continuous, then $f\left((a,b)\right)=(m,M)$ for some $m,M\in\mathbf{R}$",

(i) State whether this statement is true or false, and (ii) Prove the statement if it is true, or give a counterexample (with explicit f,a,b,m & M) if it is false.

7. a. State the *Bolzano-Weierstrass Theorem (for sets).*

b. Prove the *Bolzano-Weierstrass Theorem (for sets)* by assuming the *Completeness Axiom of* $\mathbf{R}$ *(The Least Upper Bound property of* $\mathbf{R}$*).*

We provided a proof of this theorem in class. You may give that proof or another correct proof.

8. Correct answers are +2.5 points each, wrong answers are -1 point, ambiguous answers are -2 points each, and no answers are 0 point each.

Total of problem 8 will be added to your total grade only if it is positive.

TRUE FALSE **a.** If a non-empty subset S of $\mathbf{R}$ is bounded, then it has a supremum and an infimum.

TRUE FALSE **b.** If a non-empty subset S of $\mathbf{R}$ is closed and bounded, then it has a maximum and a minimum.

TRUE FALSE **c.** Let $f : I \to \mathbf{R}$ be continuous. If I is a bounded interval, and f is a bounded function, then f must attain a maximum over I.

TRUE FALSE **d.** Every bounded sequence in $\mathbf{R}$ must have a convergent subsequence.

TRUE FALSE **e.** The set $\{0\} \cup \left\{ \frac{(-1)^n}{n} : n \in \mathbf{N} \right\}$ is not an open subset of $\mathbf{R}$,but it is a compact set in $\mathbf{R}$.

TRUE FALSE **f.** Every Cauchy sequence in $\mathbf{Q}$ (rational numbers) converges to a limit in $\mathbf{R}$.

2005

1. a. Evaluate $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ and justify your answer. (You may assume that $\lim_{x\to c} \sqrt{x} = \sqrt{c}$, for $c \geq 0$.)

b. Prove that $\lim_{x\to 2} (6x-7) = 5$ directly from the definition of the limit ($\forall \varepsilon \exists \delta$...).

2. a. State the definition of *continuity* of a function $f : I \to \mathbf{R}$ at a point $c \in I$.

b. State the definition of *differentiability* of a function $f : I \to \mathbf{R}$ at a point $c \in I$.

c. Prove the following (Theorem 25.6). Let $f : I \to \mathbf{R}$ and $c \in I$ where I is an interval in $\mathbf{R}$. If f is differentiable at c, then f is continuous at c.

3. Let $g(x) = \begin{cases} x \cdot \cos\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} : \mathbf{R} \to \mathbf{R}$.

a. Is the function $g(x)$ differentiable at $x = 0$? Prove your assertion.

b. Is the function $g(x)$ continuous at $x = 0$? Prove your assertion.

4. a. State *the Intermediate Value Theorem.*

b. Let $f : [a, b] \to [a, b]$ be continuous. Prove that there exist $c \in [a, b]$ such that $f(c) = c$.

5. a. State *the Mean Value Theorem.*

b. Let $f : (a, b) \to \mathbf{R}$ be differentiable on (a, b).

Prove that $f'(x) \leq 0$ if and only if f is decreasing on (a, b).

6. Let $S = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ be a subset of $\mathbf{R}$. In parts (**a**) and (**b**), prove your assertions by using definitions and theorems. State or indicate the theorems you use.

a. Is 1 an interior point of S? Prove your assertion.

b. Is 0 an accumulation point of S? Prove your assertion.

c. No need for explanations or proofs for this part. Circle your answer or write in your answer. If an answer is "Does not exist", then state it so.

Is S an open set? Yes or No   Is S a closed set? Yes or No

Is S a (sequentially) compact set? Yes or No

The closure of S is ____________________ The boundary of S is ____________________ The interior of S is ____________________

7. a. State *Bolzano-Weierstrass Theorem (for sets).*

b. Provide a proof of the following (part of a) theorem we proved in class.

Let S be a subset of $\mathbf{R}$ such that every sequence (s_n) in S has a convergent subsequence (s_{n_k}) whose limit is in S. Prove that S is a closed subset of $\mathbf{R}$.

8. TRUE OR FALSE. CIRCLE YOUR ANSWERS. SHOW NO WORK.

TRUE FALSE **a.** If $f : A \to \mathbf{R}$ is a bounded continuous function, then A is a bounded subset of $\mathbf{R}$.

TRUE FALSE **b.** If a set S has an accumulation point, then it must be an infinite set.

TRUE FALSE **c.** If $f : B \to \mathbf{R}$ is a function and B is a finite subset of $\mathbf{R}$, then f must attain its maximum and minimum values on B.

TRUE FALSE **d.** If (s_n) is a subsequence of (t_n) and (t_n) is a subsequence of (s_n), then $(s_n) = (t_n)$.

TRUE FALSE **e.** For every continuous function $f : \mathbf{R} \to \mathbf{R}$, if I is an interval, then $f(I)$ is an interval.

TRUE FALSE **f.** A non-empty subset C of $\mathbf{R}$ is bounded if and only if it has a supremum and an infimum.

FINAL EXAM- 2004

1. a. Determine whether the following limit exists or not. If the limit value exists, evaluate it; or if the limit does not exist, state it so. Justify your answer, and state the theorems you use. (Giving a number of a theorem does not suffice.) :

a. $\lim\limits_{x \to \infty} \sin x$

b. Prove that $\lim\limits_{x \to \infty} \frac{-1}{x+1} = 0$ directly from the definition of the limit at ∞ $(\forall \varepsilon \exists M ...)$.

2. a. Evaluate $\lim\limits_{x \to 3} \frac{x^2-9}{x-3}$ and justify your answer.

b. Prove that $\lim\limits_{x \to 2} (4x - 5) = 3$ directly from the definition of the limit $(\forall \varepsilon \exists \delta ...)$.

3. a. State the definition of *continuity* of a function $f : D \to \mathbf{R}$ at a point.

b. Prove the following (Theorem 4.1.9). Consider functions $f : A \to \mathbf{R}$ and $g : B \to \mathbf{R}$ with $A, B \subset \mathbf{R}$ such that $f(A) \subset B$. If f is continuous at some $a \in A$ and g is continuous at $b = f(a) \in B$, then the function $g \circ f$ is continuous at a.

4. a. State *the Extreme Value Theorem.*

b. Prove the following (Corollary 4.3.9).

If a function $f : [a, b] \to \mathbf{R}$ is nonconstant and continuous, then the range of f is an interval $[m, M]$ with $m, M \in \mathbf{R}$. Identify the theorems you use.

5. a. State *the Intermediate Value Theorem.*

b. Show that the equation $2x = \pi \cos x$ has a solution $x = c$ for some $c \in (0, \frac{\pi}{2})$. You may assume that $\cos x$ is continuous.

6. a. State the definition of *differentiability* of a function $f : (a, b) \to \mathbf{R}$ at a point $c \in (a, b)$

b. Prove the Product Rule, Theorem 5.2.1(b):

Let $f : (a, b) \to \mathbf{R}$ and $g : (a, b) \to \mathbf{R}$ be differentiable at $c \in (a, b)$.

Prove that fg is differentiable at c and $(fg)'(c) = f\,'(c)g(c) + f(c)g'(c)$.

7. a. State *the Mean Value Theorem.*

b. Let $f : (a, b) \to \mathbf{R}$ be differentiable on (a, b).

Prove that $f\,'(x) \geq 0$ if and only if f is increasing on (a, b).

8. TRUE OR FALSE

TRUE FALSE **a.** $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ is a closed subset of $\mathbf{R}$.

TRUE FALSE **b.** For every function $f : \mathbf{R} \to \mathbf{R} - \{0\}$, $\lim\limits_{x \to \infty} f(x) = \infty$ if and only if $\lim\limits_{x \to \infty} \frac{1}{f(x)} = 0$.

TRUE FALSE **c.** If a function is continuous over an interval and it is bounded then it must attain its maximum value.

TRUE FALSE **d.** The function $g(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentible at $x = 0$.

TRUE FALSE **e.** Every function $h : \mathbf{N} \to \mathbf{R}$ is continuous.

FINAL EXAM- 2003

1. Determine whether each the following limits exists or not. For each, if the limit value exists, calculate it; or if the limit does not exist, state it so. Justify your answers by applying theorems or by providing proofs. You may assume that $\forall x \in \mathbf{R}, |\sin x| \leq 1$.

a. $\lim_{x\to 0} \sin \frac{1}{x}$

b. $\lim_{x\to 0} x^2 \sin \frac{1}{x}$

2. Suppose that $f : (a, b) \to \mathbf{R}$ is continuous and $f(r) = 0$ for every rational number $r \in (a, b) \cap \mathbf{Q}$. Prove that $f(x) = 0$, for every $x \in (a, b)$.

3. a. State *the Intermediate Value Theorem.*

b. Let $f : [0, 1] \to [0, 1]$ be continuous. Prove that there exist $c \in [0, 1]$ such that $f(c) = c$.

4. Determine whether $h(x) = \frac{1}{x}$ is **uniformly continuous** or not on the given sets below. You may assume that h is continuous on $(0, \infty)$. Justify your answers by applying theorems or by providing proofs.

a. $D_1 = [1, 2]$

b. $D_2 = [2, \infty)$

c. $D_3 = (0, 1)$

5. Prove the Product Rule: (Since this is a theorem proved in class as well as in the textbook, you can not refer to those proofs. You are expected to provide one of these proofs, or some other correct proof below.)

Let $f : (a, b) \to \mathbf{R}$ and $g : (a, b) \to \mathbf{R}$ be differentiable at $c \in (a, b)$.
Prove that fg is differentiable at c and $(fg)'(c) = f\,'(c)g(c) + f(c)g'(c)$.

6. a. State *the Mean Value Theorem.*

b. You may assume that the derivative of $\sin x$ is $\cos x$, and both of them are bounded by 1 in absolute value. Prove that

$$\left|\frac{\sin am - \sin bm}{m}\right| \leq |a - b|\,,\ \text{if } m \neq 0.$$

7. TRUE OR FALSE.

TRUE FALSE **a.** $\left\{\frac{(-1)^n}{n} : n \in \mathbf{N}\right\}$ is a compact subset $\mathbf{R}$.

TRUE FALSE **b.** Every Cauchy sequence in $\mathbf{Q}$ is convergent to a limit in $\mathbf{R}$.

TRUE FALSE **c.** There exists a set $S \subset \mathbf{R}$ such that S has infinitely many elements, but S has no accumulation points in $\mathbf{R}$.

TRUE FALSE **d.** Let $s_n = (1 + (-1)^n)\, n^2, \forall n \in \mathbf{N}$. The sequence (s_n) has a convergent subsequence, although (s_n) is unbounded and divergent.

TRUE FALSE **e.** If $f : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ but not differentiable at 0, then $\lim_{x\to 0}\left(\frac{1}{x}\int_0^x f(t)dt\right)$ does not exist.

TRUE FALSE **f.** There does not exist any open and compact subset T of $\mathbf{R}$.

FINAL EXAM-2002

1. Determine whether or not the following limits and derivative exist. For each, calculate the value if it exists, or state it so if it does not exist. Justify your answers.

a. $\lim_{x\to 0} \left|\sin \frac{1}{x}\right|$

b. $\lim_{x\to 0} x \sin \frac{1}{x}$

c. $f\,'(0)$ where $f(x) = x^2 \sin \frac{1}{x}$ when $x \neq 0$, and $f(0) = 0$.

2. Let $f : D \to \mathbf{R}$ be a continuous function. Define $|f| : D \to \mathbf{R}$ by $|f|\,(x) = |f(x)|$.

Prove that $|f|$ is a continuous function on D.

Caution: If you want to use the continuity of the function $|x|$, then you must provide a proof of it.

3. a. State *the Intermediate Value Theorem.*

b. Let $f : [a, b] \to \mathbf{R}$ and $g : [a, b] \to \mathbf{R}$ be continuous functions such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Prove that there exist $c \in [a, b]$ such that $f(c) = g(c)$.

4. Determine whether $h(x) = \frac{1}{x^2}$ is **uniformly continuous** or not on the given sets below.

Justify your answers. You may assume that h is continuous on $(0, \infty)$

a. $D_1 = (0, 1)$

b. $D_2 = (0, \infty)$

c. $D_3 = [1, 100]$

d. $D_4 = (1, 100)$

5. Prove the Product Rule:

Let $f : (a, b) \to \mathbf{R}$ and $g : (a, b) \to \mathbf{R}$ be differentiable at $c \in (a, b)$.

Prove that fg is differentiable at c and $(fg)'(c) = f\,'(c)g(c) + f(c)g'(c)$.

6. a. State *the Mean Value Theorem.*

b. You may assume that the derivative of $\sin x$ is $\cos x$, and both of them are bounded by 1 in absolute value. Prove that

$$\left|\frac{\sin ax - \sin bx}{x}\right| \leq |a - b| \text{ if } a \neq b \text{ and } x \neq 0.$$

c. Let $F : \mathbf{R} \to \mathbf{R}$ be differentiable on $\mathbf{R}$ with $\forall x \in \mathbf{R}$, $|F'(x)| \leq 2$.

Prove that F is uniformly continuous on $\mathbf{R}$.

7. TRUE OR FALSE.

TRUE FALSE **a.** Every bounded subset S of $\mathbf{R}$ must have an infimum and a supremum.

TRUE FALSE **b.** Every Cauchy sequence in $\mathbf{Q}$ (rational numbers) is convergent to a limit in $\mathbf{Q}$.

TRUE FALSE **c.** There are infinite subsets of $\mathbf{R}$ with no accumulation points.

TRUE FALSE **d.** If a subset S of $\mathbf{R}$ is not open, then it must be closed.

TRUE FALSE **e.** If $f : \mathbf{R} \to \mathbf{R}$ is continuous at 0, then $\lim_{x\to 0} \frac{1}{x} \int_0^x f(t)dt$ exists.

TRUE FALSE **f.** For any open subset D of $\mathbf{R}$ and $F : D \to \mathbf{R}$, if F is differentiable on D and $F\,'(x) > 0$, then F^{-1} exists and it is differentiable.