Local Structure of Ideal Shapes of Knots, II
Constant Curvature Case

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Abstract. The thickness, $NIR(K)$ of a knot or link K is defined to be the radius of the largest open solid tube one can put around the curve without any self intersections of the normal discs, which is also known as the normal injectivity radius of K. For $C^{1,1}$ curves K, $NIR(K) = min\{\frac{1}{2}DCSC(K), \frac{1}{sup\kappa(K)})\}$, where $\kappa(K)$ is the generalized curvature, and the double critical self distance $DCSD(K)$ is the shortest length of the segments perpendicular to K at both end points. The knots and links in ideal shapes (or tight knots or links) belong to the minima of *ropelength = length/thickness* within a fixed isotopy class. In this article, we prove that $NIR(K) = \frac{1}{2}DCSC(K)$, for every relative minimum K of ropelength in $\mathbf{R}^n$ for certain dimensions n, including $n = 3$.

1. Introduction

In this article, the local structure of $C^{1,1}$ relatively extremal knots and links in $\mathbf{R}^n$ will be studied, particularly the extremal knots and links with maximal constant generalized curvature. The non-constant curvature case was studied in our earlier article [**6**]. The thickness or the normal injectivity radius $NIR(K, \mathbf{R}^n)$ of a knotted curve (or link) is the radius of the largest open tubular neighborhood around the curve without intersections of the normal discs. Several different notations for thickness appeared in the literature. $R(K)$ was used for thickness in Litherland-Simon-Durumeric-Rawdon [**11**] and Buck and Simon [**1**]. Gonzales and Maddocks [**8**] showed that the thickness $\eta_*(K)$ was equal to the minimum $\Delta(K)$ of ρ_G, the global radius of curvature for C^2 curves. In [**2**], Cantarella-Kusner-Sullivan defined thickness $\tau(K)$ by the infimum of the global radius of curvature and proved that it was the normal injectivity radius for $C^{1,1}$ curves.

The ideal knots are the embeddings of S^1 into $\mathbf{R}^3$, maximizing $NIR(K, \mathbf{R}^3)$ in a fixed isotopy (knot) class of fixed length. More generally, a relatively extremal knot is a relative minimum of the ropelength or isoembolic length, $\ell_e(K) = \frac{\ell(K)}{NIR(K,\mathbf{R}^n)}$ in C^1 topology, where ℓ denotes the usual length. We assume that the relative minima are taken in fixed ambient isotopy classes, since the ropelength of the curves with self intersections are not finite. The tight links and ideal knots belong to the relative minima of the ropelength. The notion of ropelength has been defined and

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studied by several authors, Litherland-Simon-Durumeric-Rawdon (called its reciprocal thickness in [**11**]), Gonzales and Maddocks [**8**], Cantarella-Kusner-Sullivan [**2**] and others. Cantarella-Kusner-Sullivan [**2**] defined ideal (thickest) knots as “tight” knots.

As J. Simon pointed out that there are physical examples (no proofs) of relatively extremal unknots in $\mathbf{R}^3$, which are not circles, and hence not ideal knots. One can construct similar physical examples for composite knots. For dimensions $n \neq 3$, every 1-dimensional knot is trivial through an isotopy of curves *of zero thickness*. At a strict relative minimum K of ropelength, one can not isotope the n-dimensional solid tube of radius $NIR(K)$ around K without increasing the length of K. Hence, one should not assume that all of the relative minima of ropelength in $\mathbf{R}^n$ (for $n \neq 3$) is the absolute minimum, that is a planar circle.

The thickness can be written in terms of the generalized curvature κ and double critical self distance $DCSD(K)$ which is the shortest length of the segments perpendicular to K at both end points. *Section 2* has the formal definitions. Thickness Formula was shown for C^2-knots in $\mathbf{R}^3$ in [**11**], and for $C^{1,1}$ knots in $\mathbf{R}^3$ by Litherland in [**10**]. Also, [**2**] *Lemma 1* proved the Thickness Formula below for $C^{1,1}$ knots and links in $\mathbf{R}^3$, since the geometric and analytic focal radii are the same in $\mathbf{R}^n$, $F_g = F_k = 1/(\sup \kappa)$ by [**6**] *Lemma 2*.

The notion of the global radius of curvature ρ_G developed by Gonzales and Maddocks for smooth curves in $\mathbf{R}^3$ defined by using circles passing through 3 points of the curve is another characterization of $NIR(K, \mathbf{R}^3)$ [**8**]. This is still true for all continuous curves by [**2**] *Lemma 1*. The construction of ρ_G and rolling ball radius R_O for curves in $\mathbf{R}^3$ are different in nature due to 3-point intersection condition versus 1-point of tangency and 1-point of intersection condition. However, at the infimum they tend towards the same quantity, $NIR(K, \mathbf{R}^3)$ [**2**].

$NIR(K, M) = R_O(K, M)$, a rolling ball/bead description of the injectivity radius in $\mathbf{R}^n$, was known by Nabutovsky [**13**] for hypersurfaces, by Buck and Simon for C^2 curves [**1**], and by Cantarella, Kusner and Sullivan [**2**] *Lemma 1*. Although the equality $NIR(K, M) = R_O(K, M)$ is generalizable to all dimensions and to Riemannian manifolds [**5**], the notion of ρ_G can not be used beyond the spaces of constant curvature.

In all of our results, the manifolds K are allowed to have several components (unless stated otherwise). If K is one dimensional, we will use $\gamma : \mathbf{D} \to K$ for a parametrization of K where $\mathbf{D}$ can be taken as a finite disjoint union of intervals and circles, S^1. All closed curves are assumed to be C^1 at the closing point.

GENERAL THICKNESS FORMULA *Theorem 1* of [**5**]:
For every complete smooth Riemannian manifold M^n and every compact $C^{1,1}$ submanifold K^k $(\partial K = \emptyset)$ of M,

$$NIR(K, M) = R_O(K, M) = \min\{F_g(K), \frac{1}{2} DCSD(K)\}.$$

THICKNESS FORMULA in $\mathbf{R}^n$ [**2**], [**10**], [**5**], [**6**]:
For every union of finitely many disjoint $C^{1,1}$ simple closed curves K in $\mathbf{R}^n$,

$$NIR(K, \mathbf{R}^n) = R_O(K, \mathbf{R}^n) = \min\{F_k(K), \frac{1}{2} DCSD(K)\}.$$

REMARK 1. *Since all problems we discuss in this article involve bounded curvature in $\mathbf{R}^n$, we rescale and take $\sup \kappa = \Lambda = 1$ to simplify our presentation.*

The main question we address in [**6**] and this article is *Problem 1* which is closely related to *Problem 2, Markov-Dubins Problem:*

Problem 1. For which relatively extremal knots and links for the ropelength functional in $\mathbf{R}^n$ does one have $NIR(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$?

Problem 2. (Markov [**12**]-Dubins [**4**]) Given p, q, v, w in $\mathbf{R}^n$, with $\|v\| = \|w\| = 1$. Classify all of the shortest curves in $\mathcal{C}(p, q; v, w)$ which is the set of all C^1 curves γ between the points p and q in $\mathbf{R}^n$ with $\gamma'(p) = v$, $\gamma'(q) = w$ and the generalized curvature $\kappa(\gamma) \leq 1 = \Lambda$.

DEFINITION 1. *A $C^{1,1}$ curve $\gamma : I = [a, b] \to \mathbf{R}^n$ is called a $CLC-$curve if there are $a \leq c \leq d \leq b$ such that (i) $\gamma([c, d])$ is a line segment of possibly zero length, and (ii) each of $\gamma([a, c])$ and $\gamma([d, b])$ is a planar circular arc of radius 1 and of length in $[0, 2\pi)$. The curve γ need not be planar. Similarly, one can define CCC-curves by C^1-concatenation of 3 arcs of circles of curvature 1, where each arc has positive length and successive arcs have different centers.*

In an earlier article [**6**], the author resolved *Problem 1* in $\mathbf{R}^n$, if the curve K did not have constant curvature, see below. The main tool in proving *Theorem 2* of [**6**] is the study of $CLC-$curves. Let $I_c(K)$ denote the minimal DCSD points on K. *Theorem 2* of [**6**] tells us that the sections of a relatively extremal knot in $\mathbf{R}^n$ with $I_c(K)$ removed are $CLC-$curves or overwound, i.e. $\kappa \equiv \Lambda$. This generalizes one of the earliest results about the shape of ideal knots, that was obtained by Gonzales and Maddocks [**8**] p 4771: *A smooth ideal knot can be partitioned into arcs of constant (maximal) global curvature and line segments.*

Durumeric [**6**] *Theorem 2:*

Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$ and $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ be a parametrization. If K is a relative minimum for ℓ_e and $\exists s_0 \in \mathbf{D}$, $\kappa(\gamma)(s_0) < \sup \kappa(\gamma)$, then both of the following hold for $\gamma(\mathbf{D}) = K$:

(i) $NIR(K, \mathbf{R}^n) = R_O(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$.

(ii) If $s_0 \notin I_c(K)$, then there exists a, b such that $s_0 \in [a, b]$, $\gamma([a, b])$ is a $CLC(\sup \kappa(\gamma))$-curve where the line segment has positive length and contains $\gamma(s_0)$, and each circular part has at most π radians angle ending at a point of $I_c(K)$.

Durumeric [**6**] *Corollary 2: Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$. If K is a relative minimum for ℓ_e and curvature of K is not identically constant $R_O(K)^{-1}$, then the thickness of K is $\frac{1}{2}DCSD(K)$. Equivalently, if there exists a relative minimum K for ℓ_e such that $\frac{1}{2}DCSD(K) > R_O(K) = F_k(K)$, then K must have constant generalized curvature $F_k(K)^{-1}$.*

Markov [**12**], Dubins [**4**] and Reeds and Shepp [**14**] studied the 2$-$dimensional cases for *Problem 2*. In dimension 3, the following results of H. Sussmann obtain the possible types solutions for this problem. A helicoidal arc is a smooth curve in $\mathbf{R}^3$ with constant curvature 1 and positive torsion τ satisfying the differential equation $\tau'' = 1.5\tau'\tau^{-1} - 2\tau^3 + 2\tau - \zeta\tau\,|\tau|^{1/2}$ for some nonnegative constant ζ.

Sussmann [**15**] *Theorems:*

1. For the Markov-Dubins problem in dimension three, every minimizer is either (a) a helicoidal arc or (b) a concatenation of three pieces each of which is a circle or a straight line. For a minimizer of the form CCC, the middle arc has length $\geq \pi$ and $< 2\pi$.

2. Every helicoidal arc corresponding to a value of ζ such that $\zeta > 0$ is local strict minimizer.

Sussmann further proves that *CSC-conjecture* (every minimizer is either *CCC* or *CLC,* [**14**]) is false in $\mathbf{R}^3$: [**15**] *Propositions 2.1* and *2.2.* In [**15**], the details of the steps of the proof of *Theorem 1* (of Sussmann) are provided, but there are only few remarks about proof of its *Theorem 2.*

The main result of this article is *Theorem 1* below which shows the nonexistence of a relative minimum K for ℓ_e with $\frac{1}{2}DCSD(K) > F_k(K)$, in all cases in certain dimensions, including constant curvature cases. Only *Corollary 1* invokes *Theorem 1 of Sussmann* [**15**]. The remaining results and proofs of this article including *Theorem 1* and [**6**] are independent of Sussmann [**15**], especially since they are not restricted to dimension 3.

THEOREM 1. *Let n be a dimension such that (i) every minimizer for the Markov-Dubins problem in $\mathbf{R}^n$ is either a smooth curve with curvature 1 and positive torsion, or a C^1-concatenation of finitely many circular arcs of curvature* 1 *and a line segment, and (ii) every $CCC-$curve with the middle arc of length $< \pi$ is not a minimizer. Then, $NIR(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$ for every relative minimum K of ℓ_e where K is a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$.*

COROLLARY 1. *Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^3$. If K is a relative minimum of ℓ_e, then $NIR(K, \mathbf{R}^3) = \frac{1}{2}DCSD(K)$.*

2. Definitions and Notation

We assume all parametrizations $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ are one-to-one and $\|\gamma'\| \neq 0$.

DEFINITION 2. *$\exp_p^N v = p + v : NK \to \mathbf{R}^n$ is the normal exponential map of K in $\mathbf{R}^n$. The thickness of K in $\mathbf{R}^n$ or the normal injectivity radius of $\exp^N$ is*

$$NIR(K, \mathbf{R}^n) = \sup(\{0\} \cup \{r > 0 : \exp^N : \{v \in NK : \|v\| < r\} \to \mathbf{R}^n \text{ is one-to-one}\}).$$

Equivalently, if $\gamma(s)$ parametrizes K, then

$$r > NIR(K, \mathbf{R}^n) \Leftrightarrow \left(\begin{array}{c} \exists \gamma(s), \gamma(t), q \in \mathbf{R}^n, \\ \gamma(s) \neq \gamma(t), \|\gamma(s) - q\| < r, \|\gamma(t) - q\| < r, \text{ and} \\ (\gamma(s) - q) \cdot \gamma'(s) = (\gamma(t) - q) \cdot \gamma'(t) = 0 \end{array} \right).$$

DEFINITION 3. *For $\gamma : I \to \mathbf{R}^n$, define:*

$$\textit{Dilation} : dil^\alpha \gamma'(s,t) = \frac{\measuredangle(\gamma'(s), \gamma'(t))}{\ell(\gamma([s,t]))} \text{ for } s \neq t$$

$$\textit{(Generalized) Curvature} : \kappa\gamma(s) = \kappa(\gamma)(s) = \limsup_{t \neq u \text{ and } t,u \to s} dil^\alpha \gamma'(t,u)$$

$$\textit{Analytic focal distance} : F_k(\gamma) = (\sup_I \kappa\gamma(s))^{-1}.$$

If K is a union of finitely many disjoint $C^{1,1}$ curves $\gamma_{(i)}$ in $\mathbf{R}^n$, then $F_k(K) = \min_i F_k(\gamma_{(i)})$.

Definition 4. *Let* K *be a finite union of disjoint* C^1 *curves in* $\mathbf{R}^n$*. For any* $v \in UT\mathbf{R}^n_p$ *and any* $r > 0$*, define*

(i) $O_p(v,r) = \{x \in \mathbf{R}^n : \exists w \in \mathbf{R}^n, v \cdot w = 0, \|w\| = 1, \|x - p - rw\| < r\}$
(ii) $O^c_p(v,r) = \mathbf{R}^n - O_p(v,r)$
(iii) $O_p(r;K) = O_p(v,r)$ *where* $v \in UTK_p$
(iv) $O(r;K) = \bigcup_{p \in K} O_p(r;K)$

In all of the above, r may be omitted when $r = 1$. K will be omitted unless there is an ambiguity.

Definition 5. *Let* K *be a finite union of* C^1 *curves in* $\mathbf{R}^n$*. Define*
(i) The ball radius of K *in* $\mathbf{R}^n$ *to be* $R_O(K,\mathbf{R}^n) = \inf\{r > 0 : O(r;K) \cap K \neq \emptyset\}$
(ii) The pointwise geometric focal distance to be
$F_g(p) = \inf\{r > 0 : p \in \overline{O_p(r;K) \cap K}\}$ *for any* $p \in K$,
and the geometric focal distance to be $F_g(K) = \inf_{p \in K} F_g(p)$*.*

Definition 6. *A pair of distinct points* p *and* q *in* K *are called a double critical pair for* K*, if the line segment* $\overline{pq}$ *is normal to* K *at both* p *and* q*. The double critical self distance is*

$$DCSD(K) = \inf\{\|p - q\| : \{p,q\} \text{ is a double critical pair for } K\}.$$

A double critical pair $\{p,q\}$ *is called minimal if* $DCSD(K) = \|p - q\|$*.*

3. Review of Some Basic Tools

K denotes a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$ and $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ denotes a one-to-one non-singular parametrization, where $\mathbf{D} = \bigcup_{i=1}^k \mathbf{S}^1_{(i)}$, a union of k copies of disjoint circles, unless stated otherwise. When $\|\gamma'\| \equiv 1$ is assumed, $\mathbf{S}^1_{(i)}$ are taken with the appropriate radius and length. A knot or link class $[\theta]$ is a free C^1 (ambient) isotopy class of embeddings of $\gamma : \mathbf{D} \to \mathbf{R}^n$ with a fixed number of components. Since all of our proofs involve local perturbations of only one component at a time, we will work with $\gamma_{(i)} : \mathbf{S}^1_{(i)} \to \mathbf{R}^n$ and we will omit the lower index $_{(i)}$ to simplify the notation wherever it is possible. We will identify $\mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z}$, for $L > 0$, and use interval notation to describe connected proper subsets of $\mathbf{R}/L\mathbf{Z}$. In other words, $\gamma_{(i)}(t + L) = \gamma_{(i)}(t)$ and $\gamma'_{(i)}(t + L) = \gamma'_{(i)}(t)$, $\forall t \in \mathbf{R}$ with $\left\|\gamma'_{(i)}\right\| \neq 0$ and $\gamma_{(i)}$ is one-to-one on $[0, L)$. See [**6**] Section 4 for proofs of the following propositions that will be used in this article.

Definition 7. *For any* $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$*, one defines the ropelength or extrinsically isoembolic length to be* $\ell_e(\gamma) = \ell_e(K) = \frac{\ell(K)}{R_o(K)} = \frac{vol_1(K)}{NIR(K,\mathbf{R}^n)}$*.*

Lemma 1. *Let* $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ *be a* C^1 *knot or link. (i) If* $DCSD(K) > 0$*, then there exists a critical pair* $\{p_0, q_0\}$ *such that* $DCSD(K) = \|p_0 - q_0\|$*. (ii) If* $\sup \kappa\gamma < \infty$*, i.e.* γ *is* $C^{1,1}$*, then* $DCSD(K) > 0$.

Proposition 1. *Let* $\{\gamma_m\}_{m=1}^{\infty} : \mathbf{D} \to \mathbf{R}^n$ *be a sequence uniformly converging to* γ *in* C^1 *sense, i.e.* $(\gamma_m(s), \gamma_m'(s)) \to (\gamma(s), \gamma'(s))$ *uniformly on* $\mathbf{D}$. *Let* $K_m = \gamma_m(\mathbf{D})$ *for* $m \geq 1$ *and* $K = \gamma(\mathbf{D})$.

*(i) ([**2**] Lemma 3, and [**10**]) If* $R_O(K_m) \geq r$ *for sufficiently large* m, *then* $R_O(K) \geq r$. *Consequently,* $\limsup_m R_O(K_m) \leq R_O(K)$.

(ii) If $\liminf_m DCSD(K_m) > 0$, *then* $\liminf_m DCSD(K_m) \geq DCSD(K)$.

Definition 8. *For* $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$, *(with* $\gamma' \neq 0$*) define*

$$
\begin{aligned}
I_c &= \left\{ \begin{array}{c} x \in \mathbf{D} : \exists y \in \mathbf{D} \text{ such that } \|\gamma(x) - \gamma(y)\| = DCSD(K) \text{ and} \\ (\gamma(x) - \gamma(y)) \cdot \gamma'(x) = (\gamma(x) - \gamma(y)) \cdot \gamma'(y) = 0\} \end{array} \right\} \\
I_z &= \{x \in \mathbf{D} : \kappa\gamma(x) = 0\} \\
I_{mx} &= \{x \in \mathbf{D} : \kappa\gamma(x) = 1/R_O(K)\} \\
I_b &= \{x \in \mathbf{D} : 0 < \kappa\gamma(x) < 1/R_O(K)\}
\end{aligned}
$$

Let $K_c = \gamma_c = \gamma(I_c)$, $K_z = \gamma_z = \gamma(I_z)$, $K_{mx} = \gamma_{mx} = \gamma(I_{mx})$ *and* $K_b = \gamma_b = \gamma(I_b)$.

Proposition 2. *([**2**] Theorem 7, [**7**], [**9**]) For any knot/link class* $[\theta]$ *in* $\mathbf{R}^n$, $\exists \gamma_0 \in [\theta]$ *such that*

(i) $\forall \gamma \in [\theta]$, $0 < \ell_e(\gamma_0) \leq \ell_e(\gamma)$, *and hence*

(ii) $\forall \gamma \in [\theta]$, $(\ell(\gamma_0) = \ell(\gamma) \Longrightarrow R_O(\gamma_0) \geq R_O(\gamma))$.

Proposition 3. *Let* $\{\gamma_m\}_{m=1}^{\infty} : \mathbf{D} \to \mathbf{R}^n$ *be a sequence uniformly converging to* γ *in* C^1 *sense,* $K = \gamma(\mathbf{D})$ *and* $K_m = \gamma_m(\mathbf{D})$, *such that* $\exists C < \infty, \forall m, \sup \kappa\gamma_m \leq C$.

(i) Let $A \subset D$ *be a given compact set with* $\{s \in D : \gamma_m(s) \neq \gamma(s)\} \subset A, \forall m$. *If* $A \cap I_c = \emptyset$, *then* $\exists m_1$ *such that* $\forall m \geq m_1, DCSD(K_m) \geq DCSD(K)$.

(ii) If $F_k(K) < \frac{1}{2}DCSD(K)$ *and* $F_k(K_m) \geq F_k(K), \forall m$, *then* $\exists m_1$ *such that* $\forall m \geq m_1, R_O(K_m) \geq R_O(K)$.

Proposition 4. *(Also see [**8**] p4771 for another version for smooth ideal knots.) Let* K *be a* $C^{1,1}$ *relatively minimal knot or link for the ropelength* ℓ_e.

(i) If $DCSD(K) = 2R_O(K)$, *then* $K - (K_c \cup K_{mx})$ *is a countable union of open ended line segments, and hence* $I_b \subset I_c$.

(ii) If $DCSD(K) > 2R_O(K)$, *then* $K - K_{mx}$ *is a countable union of open ended line segments, (in fact* $\emptyset$ *by* Theorem 2 *of* [**6**]*).*

4. Proof of Theorem 1

In dimension 3, the following coincides with the standard definitions except the sign of the torsion. For a C^3 curve $\gamma : I \to \mathbf{R}^n$, $n \geq 3$, parametrized by arclength, i.e. $\|\gamma'(t)\| = 1$, define

(i) $\mathbf{T} = \gamma'(t)$,

(ii) $\kappa = \|\mathbf{T}'\|$, and if $\kappa > 0$, define $\mathbf{N} = \frac{1}{\kappa}\mathbf{T}'$,

(iii) $\tau = \|\mathbf{N}' + \kappa\mathbf{T}\| \geq 0$, and if $\tau > 0$, define $\mathbf{B} = \frac{1}{\tau}(\mathbf{N}' + \kappa\mathbf{T})$.

This definition yields an orthonormal set $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along γ, if both $\kappa, \tau > 0$.

Proposition 5. *Let* K *be a union of finitely many disjoint* $C^{1,1}$ *simple closed curves in* $\mathbf{R}^n$ *and* K *be a relative minimum of* ℓ_e. *If* K *has a component* K_0 *which is a* C^4 *simple closed curve of positive torsion* $\tau > 0$ *everywhere, then* $NIR(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$.

PROOF. (i) First prove the statement for a connected K. Let $\gamma : \mathbf{S}^1 \to K \subset \mathbf{R}^n$ parametrize K. Proposition holds if curvature of γ is not identically $NIR(K, \mathbf{R}^n)^{-1}$, by [**6**] *Corollary 2*. By rescaling, assume that $\kappa\gamma \equiv NIR(K, \mathbf{R}^n)^{-1} = 1$.

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B} = -\mathbf{T} + \tau\mathbf{B}$$

$$\mathbf{T} \cdot \mathbf{N}' = -\mathbf{T}' \cdot \mathbf{N} = -\kappa = -1$$

$$\mathbf{N} \cdot \mathbf{N}'' = \frac{1}{2}(\mathbf{N} \cdot \mathbf{N})'' - \mathbf{N}' \cdot \mathbf{N}' = -\left\|-\mathbf{T} + \tau\mathbf{B}\right\|^2 = -(1 + \tau^2) \tag{4.1}$$

Consider the variation $\gamma_\varepsilon(t) = \gamma(t) + \varepsilon\mathbf{N}(t)$ and $\Gamma_\varepsilon(t) = \frac{L}{\ell(\gamma_\varepsilon)}\gamma_\varepsilon(t)$. By *Lemma 2* below,

$$\frac{d}{d\varepsilon}\kappa\Gamma_\varepsilon\Big|_{\varepsilon=0}(t) = \gamma''(t) \cdot \mathbf{N}''(t) - 2\gamma'(t) \cdot \mathbf{N}'(t) + \frac{1}{L}\int_0^L \gamma'(u) \cdot \mathbf{N}'(u)du$$

$$= \mathbf{N} \cdot \mathbf{N}''(t) - 2\mathbf{T} \cdot \mathbf{N}'(t) + \frac{1}{L}\int_0^L \mathbf{T} \cdot \mathbf{N}'(u)du$$

$$= -(1 + \tau^2) + 2 + \frac{1}{L}\int_0^L -1du \tag{4.2}$$

$$\frac{d}{d\varepsilon}\kappa\Gamma_\varepsilon\Big|_{\varepsilon=0}(t) = -\tau^2(t) < 0 \tag{4.3}$$

Since K is compact and $\kappa\Gamma_\varepsilon$ is a C^2 function of t and ε, there exists $\varepsilon_0 > 0$ such that $\forall\varepsilon \in (0, \varepsilon_0), \forall t \in \mathbf{S}^1, (\kappa\Gamma_\varepsilon(t) < 1)$. Hence, $\forall\varepsilon \in (0, \varepsilon_0)$, $\max\kappa\Gamma_\varepsilon < 1$ and $F_k(\Gamma_\varepsilon) > 1 = F_k(K)$. Obviously, $\Gamma_{1/m} \to \gamma$ in C^1 sense, as $m \to \infty$. Suppose that $\frac{1}{2}DCSD(K) > F_k(K)$. Let $K_m = \Gamma_{1/m}(\mathbf{S}^1)$. *By Proposition 1(ii)*,

$$\liminf_m \frac{1}{2}DCSD(K_m) \geq \frac{1}{2}DCSD(K) > F_k(K) = 1.$$

For sufficiently large m,

$$\frac{1}{2}DCSD(K_m) > 1$$

$$F_k(K_m) > 1$$

$$NIR(K_m, \mathbf{R}^n) > 1 = F_k(K) = NIR(K, \mathbf{R}^n) \text{ (Thickness Formula)}$$

$$\ell_e(K_m) < \ell_e(K), \text{ since } \ell(K_m) = L = \ell(K)$$

This contradicts to the fact that K is relative minimum of ℓ_e.

Consequently, $\frac{1}{2}DCSD(K) \leq F_k(K)$, that is $NIR(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$.

(ii) To prove this statement for K with two or more components, let K_0 be a C^4 simple closed curve component of positive torsion $\tau > 0$ everywhere. By *Corollary 2* of [**6**], the only remaining case is when all of K has constant curvature 1. Let $\gamma : \mathbf{S}^1 \to K_0 \subset \mathbf{R}^n$, and take the variation and rescaling $\Gamma_\varepsilon(t) = \frac{\ell(K_0)}{\ell(\gamma_\varepsilon)}\gamma_\varepsilon(t)$ **only** along K_0 and leave the other components invariant. Let $K_m^* = \Gamma_{1/m}(\mathbf{S}^1) \cup (K - K_0)$. Obviously, $K_m^* \to K$ in C^1 sense. Repeat the same proof as in (i) until and including "Suppose that $\frac{1}{2}DCSD(K) > F_k(K)$". By *Proposition 1(ii)*,

$$\liminf_m \frac{1}{2}DCSD(K_m^*) \geq \frac{1}{2}DCSD(K) > F_k(K) = 1.$$

For sufficiently large m,

$$\begin{aligned}
\frac{1}{2}DCSD(K_m^*) &> 1 \\
F_k(K_m^*) &= F_k(K-K_0) = 1 < F_k(\Gamma_{1/m}) \\
NIR(K_m^*, \mathbf{R}^n) &= 1 = F_k(K) = NIR(K, \mathbf{R}^n) \text{ (Thickness Formula)} \\
\ell_e(K_m^*) &= \ell_e(K), \text{ since } \ell(K_m^*) = L = \ell(K_0)
\end{aligned}$$

Since K is a relative minimum of ℓ_e, K_m^* is a relative minimum of ℓ_e, for sufficiently large m. K_m^* does not have constant maximal curvature 1 everywhere, since $\max \kappa\Gamma_{1/m} < 1$. Hence, by [**6**] *Corollary 2,* $1 = NIR(K_m^*, \mathbf{R}^n) = \frac{1}{2}DCSD(K_m^*)$ for sufficiently large m, which contradicts above. Consequently, $\frac{1}{2}DCSD(K) \leq F_k(K)$, that is $NIR(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$. □

LEMMA 2. *Let $\gamma : [0, L] \to \mathbf{R}^n$ be be a C^2-curve parametrized by arclength with constant curvature 1: $\|\gamma'(t)\| = \|\gamma''(t)\| = 1$ and let $V : [0, L] \to \mathbf{R}^n$ be a C^2 vector field along γ normal to γ, i.e. $V(t) \cdot \gamma'(t) = 0$. Define $\gamma_\varepsilon(t) = \gamma(t) + \varepsilon V(t)$ and $\Gamma_\varepsilon(t) = \frac{L}{\ell(\gamma_\varepsilon)}\gamma_\varepsilon(t)$, where $L = \ell(\gamma)$. Then,*

$$\frac{d}{d\varepsilon}\kappa\gamma_\varepsilon(t)\bigg|_{\varepsilon=0} = \gamma''(t) \cdot V''(t) - 2\gamma'(t) \cdot V'(t) \text{ and} \tag{4.4}$$

$$\frac{d}{d\varepsilon}\kappa\Gamma_\varepsilon(t)\bigg|_{\varepsilon=0} = \gamma''(t) \cdot V''(t) - 2\gamma'(t) \cdot V'(t) + \frac{1}{L}\int_0^L \gamma'(u) \cdot V'(u)du \tag{4.5}$$

PROOF. We include this elementary computation for the sake of completeness. Recall that:

$$\kappa\alpha = \|\alpha''(\alpha' \cdot \alpha') - \alpha'(\alpha'' \cdot \alpha')\| \, \|\alpha'\|^{-4} \tag{4.6}$$

$$\frac{d}{d\varepsilon}\left\|v_0 + \varepsilon v_1 + \varepsilon^2 v_2\right\|^k\bigg|_{\varepsilon=0} = k(v_0 \cdot v_1)\,\|v_0\|^{k-2} \tag{4.7}$$

$$\begin{aligned}
\gamma'' \cdot \gamma' &= 0 \\
\|\gamma'\| = \|\gamma''\| &= \kappa\gamma = 1 \\
\gamma_\varepsilon'(t) &= \gamma'(t) + \varepsilon V'(t) \\
\gamma_\varepsilon''(t) &= \gamma''(t) + \varepsilon V''(t)
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\gamma_\varepsilon''(\gamma_\varepsilon' \cdot \gamma_\varepsilon') - \gamma_\varepsilon'(\gamma_\varepsilon'' \cdot \gamma_\varepsilon') &= \gamma'' + \varepsilon\left[V'' + 2\gamma''(V' \cdot \gamma') - \gamma'(V'' \cdot \gamma') - \gamma'(\gamma'' \cdot V')\right] + o(\varepsilon^2) \\
&:= w_0 + \varepsilon w_1 + o(\varepsilon^2)
\end{aligned} \tag{4.9}$$

$$w_0 \cdot w_1 = \gamma'' \cdot [V'' + 2\gamma''(V' \cdot \gamma') - \gamma'(V'' \cdot \gamma') - \gamma'(\gamma'' \cdot V')] = \gamma'' \cdot V'' + 2(V' \cdot \gamma') \tag{4.10}$$

$$\begin{aligned}
\frac{d}{d\varepsilon}\kappa\gamma_\varepsilon\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\|\gamma_\varepsilon''(\gamma_\varepsilon' \cdot \gamma_\varepsilon') - \gamma_\varepsilon'(\gamma_\varepsilon'' \cdot \gamma_\varepsilon')\| \, \|\gamma_\varepsilon'\|^{-4}\bigg|_{\varepsilon=0} \\
&= (\gamma'' \cdot V'' + 2(V' \cdot \gamma'))\,\|\gamma''\|^{-1}\,\|\gamma'\|^{-4} - 4\,\|\gamma''\|\,\|\gamma'\|^{-6}\,(V' \cdot \gamma') \\
&= \gamma'' \cdot V'' + 2V' \cdot \gamma' - 4V' \cdot \gamma' = \gamma'' \cdot V'' - 2V' \cdot \gamma'
\end{aligned} \tag{4.11}$$

This proves (4.4).

$$\Gamma_\varepsilon(t) = \frac{L}{\ell(\gamma_\varepsilon)}\gamma_\varepsilon(t) \text{ hence } \kappa\Gamma_\varepsilon(t) = \frac{\ell(\gamma_\varepsilon)}{L}\kappa\gamma_\varepsilon(t) \tag{4.12}$$

By using the usual First Variation Formula [**3**], we obtain (4.5):

$$\begin{aligned}
\frac{d}{d\varepsilon}\ell(\gamma_\varepsilon)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_0^L \|\gamma_\varepsilon'(u)\|\,du = \int_0^L \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\|\gamma_\varepsilon'(u)\|\,du \\
&= \int_0^L \|\gamma'\|^{-1} V'\cdot\gamma' du = \int_0^L V'\cdot\gamma' du \qquad (4.13) \\
\frac{d}{d\varepsilon}\kappa\Gamma_\varepsilon(t)\Big|_{\varepsilon=0} &= \frac{\ell(\gamma)}{L}\cdot\frac{d}{d\varepsilon}\kappa\gamma_\varepsilon(t)\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon}\frac{\ell(\gamma_\varepsilon)}{L}\Big|_{\varepsilon=0}\cdot\kappa\gamma(t) \\
&= \left(\gamma''\cdot V'' - 2V'\cdot\gamma'\right)(t) + \frac{1}{L}\int_0^L V'\cdot\gamma' du \qquad (4.14)
\end{aligned}$$

□

LEMMA 3. *If there exists a $C^{1,1}$ K parametrized by $\gamma : \mathbf{S}^1 \to K \subset \mathbf{R}^n$ satisfying both (i) K is a relative minimum of ℓ_e in $\mathbf{R}^n$, and (ii) $NIR(K,\mathbf{R}^n) = F_k(K) = 1 < \frac{1}{2}DCSD(K)$, then $\exists\delta > 0$ such that $\gamma([a,b])$ is a shortest curve in $\mathcal{C}(\gamma(a),\gamma(b);\gamma'(a),\gamma'(b))$ whenever $\ell_{ab}(\gamma) \le \delta$. This is also true for K with several components.*

PROOF. By [**6**] *Corollary 2*, $\kappa\gamma \equiv 1$. Reparametrize $\gamma : \mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z} \to K \subset \mathbf{R}^n$, such that $\|\gamma'\| = \|\gamma''\| = 1$, almost everywhere. Suppose that such $\delta > 0$ does not exist. $\forall m \in \mathbf{N}, \exists a_m, b_m \in \mathbf{S}^1$ such that $0 < |a_m - b_m| \le \frac{1}{m}$ but $\gamma([a_m,b_m])$ is not a shortest curve in $\mathcal{C}(\gamma(a_m),\gamma(b_m);\gamma'(a_m),\gamma'(b_m))$. By *Proposition 3* of [**6**], there exists a shortest curve θ_m in $\mathcal{C}(\gamma(a_m),\gamma(b_m);\gamma'(a_m),\gamma'(b_m))$. The C^1 end point data of θ_m and $\gamma([a_m,b_m])$ match. Let γ_m be the C^1 curve obtained from γ by removing $\gamma([a_m,b_m])$ and attaching θ_m in its place. Then, $\kappa\gamma_m \le 1$, and $L - \frac{1}{m} \le \ell(\gamma_m) < L = \ell(\gamma)$. Hence, it is possible to reparametrize γ_m uniformly with a common domain $\mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z}$, with $\|\gamma_m'\| \le 1$ and $\|\gamma_m''\| \le 1$, almost everywhere, for sufficiently large m. Hence, the convergence $\gamma_m \to \gamma$ can be taken uniformly in C^1-sense, by Arzela-Ascoli Theorem and taking a subsequence if it is necessary. Let $K_m = \gamma_m(\mathbf{S}^1)$. By *Proposition 1(ii)*,

$$\liminf_m \frac{1}{2}DCSD(K_m) \ge \frac{1}{2}DCSD(K) > F_k(K) = 1.$$

For sufficiently large m,

$$\begin{aligned}
\frac{1}{2}DCSD(K_m) &> 1 \\
F_k(K_m) &= F_k(K) = 1 \\
NIR(K_m,\mathbf{R}^n) &= NIR(K,\mathbf{R}^n) = 1 \\
\ell(K_m) &< L = \ell(K) \\
\ell_e(K_m) &< \ell_e(K)
\end{aligned}$$

which contradicts relative minimality of K for ℓ_e. If K has finitely many components, then at least one of the components of K contains infinitely many pairs $\{\gamma(a_m),\gamma(b_m)\}$ specified as above, and the rest of the proof is the same. □

Proof. (**Theorem 1**) Let n be a dimension such that (i) every minimizer for the Markov-Dubins problem in $\mathbf{R}^n$ is either a smooth curve with curvature 1 and positive torsion, or a C^1-concatenation of finitely many circular arcs of curvature 1 and a line segment, and (ii) every $CCC-$curve with the middle arc of length $< \pi$ is not a minimizer.

First consider the case of a connected K. Suppose that there exists a relative minimum K of ℓ_e such that $NIR(K, \mathbf{R}^n) = F_k(K) < \frac{1}{2}DCSD(K)$. Rescale to obtain $F_k(K) = 1$. By [**6**] *Corollary 2*, K has constant generalized curvature $\kappa = 1$. By *Lemma 3*, $\exists \delta > 0$ such that $\forall a \in \mathbf{S}^1$, $\gamma([a, a+\delta])$ is a shortest curve in $\mathcal{C}(\gamma(a), \gamma(a+\delta); \gamma'(a), \gamma'(a+\delta))$, where $\gamma : \mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z} \to K \subset \mathbf{R}^n$ is a parametrization with respect to arclength. By the hypothesis (i), each $\gamma([a, a+\delta])$ is either (a) a smooth curve with $\kappa = 1$ and $\tau > 0$ or (b) a C^1-concatenation of finitely many pieces each of which is an arc of a circle or a line segment. Since $\kappa = 1$, there are no line segments. Type (a) curves and type (b) curves do not have any curve in common, even in part, since one type has $\tau > 0$ everywhere and the other one has concatenations of planar arcs of circles. $\gamma([a, a+\delta])$ and $\gamma([a+\frac{\delta}{2}, a+\frac{3\delta}{2}])$ have a common piece, and hence they must be of the same type. Inductively, we conclude that either all of K is a smooth curve with positive torsion or a C^1-concatenation of finitely many circular arcs. *Proposition 5* and $F_k(K) < \frac{1}{2}DCSD(K)$ exclude the smooth case with $\tau > 0$, and imply that K must be a concatenation of finitely many circular arcs.

We will assume that two successive circular arcs have distinct centers, i.e. no trivial concatenations. If any of the circular arcs of K has length π or more, one can find 2 diametrically opposed points on it, forming a minimal double critical pair, and $NIR(K, \mathbf{R}^n) = F_k(K) = \frac{1}{2}DCSD(K) = 1$ contradicting the initial assumption.

This leaves us the final case of C^1-concatenations with circular arcs of length $< \pi$. There must be at least 3 circular arcs in K. Consider a parametrization $\gamma : \mathbf{S}^1 \to K$ with respect to arclength such that $\gamma([0, a])$ is a single maximal circular arc of length $a < \pi$. For m sufficiently large, $\gamma([-\frac{1}{m}, a+\frac{1}{m}])$ is a CCC-curve such that the middle arc has length $a < \pi$. By the hypothesis (ii), this type CCC-sections of K are not minimizers in a corresponding $\mathcal{C}$. Let $\mathcal{U}$ be an open set in C^1 topology such that $\gamma \in \mathcal{U}$ and $\ell_e(\gamma) \leq \ell_e(\eta), \forall \eta \in \mathcal{U} \cap [\gamma]$. When one replaces a non-minimal CCC-section with a minimal curve in the same $\mathcal{C}$, then a priori one can not assume that the new curve is in $\mathcal{U} \cap [\gamma]$, and one can not use relative minimality of K. Let θ_m be any minimizer of $\mathcal{C}(\gamma(-\frac{1}{m}), \gamma(a+\frac{1}{m}); \gamma'(-\frac{1}{m}), \gamma'(a+\frac{1}{m}))$. Let γ_m be the C^1 curve obtained from γ by removing $\gamma([-\frac{1}{m}, a+\frac{1}{m}])$ and attaching θ_m in its place. Then, $\kappa\gamma_m \leq 1$ and $L - (a+\frac{2}{m}) \leq \ell(\gamma_m) < L = \ell(\gamma)$ where $L - a > 0$ and $L < \infty$. Hence, for all sufficiently large m, it is possible to reparametrize γ_m with a common domain $\mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z}$, such that:

1. $\gamma_m(-\frac{1}{m}) = \gamma(-\frac{1}{m})$,
2. $\|\gamma_m'\| = 1$, and $\|\gamma_m''\| \leq 1$ almost everywhere, on $[-\frac{1}{m}, a+\frac{1}{m}]$, and
3. $\|\gamma_m'\| \leq c_1 < \infty$, and $\|\gamma_m''\| \leq c_2 < \infty$ almost everywhere, on $\mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z}$.

Observe that $(\|\gamma'(s) - \gamma'(t)\| \leq C\,|s-t|\,, \forall s, t \in I)$ if and only if $\|\gamma''(s)\| \leq C$ for almost all $s \in I$, and γ' is absolutely continuous. By Arzela-Ascoli Theorem, there exists a convergent subsequence (using the same indices), $\gamma_m \to \gamma_0$ converging uniformly on $\mathbf{S}^1$ in the C^1-sense.

4. $\|\gamma_0'\| = 1$ on $[0, a]$,
5. $\kappa\gamma_0 \leq 1$ on $[0, a]$, since the inequality $\|\gamma_m'(s) - \gamma_m'(t)\| \leq |s - t|$ carries to the limit: $\|\gamma_0'(s) - \gamma_0'(t)\| \leq |s - t|$,
6. $\gamma_0(0) = \gamma(0)$, and hence $\gamma_0'(0) = \gamma'(0)$, and
7. Since $\gamma_m(t_m) = \gamma(a + \frac{1}{m})$ for some $t_m \in (-\frac{1}{m}, a + \frac{1}{m})$, $\exists t_0 \in [0, a]$ such that $\gamma_0(t_0) = \gamma(a)$, and hence $\gamma_0'(t_0) = \gamma'(a)$.

By *Proposition 1* of [**6**], the planar circular arc $\gamma([0, a])$ is the unique minimizer in $\mathcal{C}(\gamma(0), \gamma(a); \gamma'(0), \gamma'(a))$ which also contains $\gamma_0([0, t_0])$.

$$a = \ell(\gamma[0, a]) \leq \ell(\gamma_0[0, t_0]) = t_0 = \lim(t_m + \frac{1}{m}) \leq \lim(a + \frac{2}{m}) = a$$

Consequently, $a = t_0$, $\gamma|[0, a] = \gamma_0|[0, a]$ and $\gamma(\mathbf{S}^1) = \gamma_0(\mathbf{S}^1) = K$. Let $K_m = \gamma_m(\mathbf{S}^1)$. By *Proposition 1(ii)*,

$$\liminf_m \frac{1}{2} DCSD(K_m) \geq \frac{1}{2} DCSD(K) > F_k(K) = 1.$$

For sufficiently large m,

$$\begin{aligned} \gamma_m &\in \mathcal{U} \cap [\gamma_0] = \mathcal{U} \cap [\gamma] \\ \frac{1}{2} DCSD(K_m) &> 1 \\ F_k(K_m) &= F_k(K) = 1 \\ NIR(K_m, \mathbf{R}^n) &= NIR(K, \mathbf{R}^n) = 1 \\ \ell(K_m) &< L = \ell(K) \\ \ell_e(K_m) &< \ell_e(K) \end{aligned}$$

which contradicts relative minimality of K. This shows the nonexistence of concatenations only with circular arcs of length $< \pi$. Actually, the existence of one circular arc of length $< \pi$ actually led to the contradiction. Since all cases lead to a contradiction, one must have $NIR(K, \mathbf{R}^n) = \frac{1}{2} DCSD(K)$. The extension to several component case is straightforward, by *Proposition 5*, *Corollary 2* of [**6**], and the proof of the final case being a local argument. □

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