PROBLEM SET 9 – §3.1

Exercise 3.1.6: Recall that Euler's formula tells us that if $\mathbf{u}_1, \mathbf{u}_2 \in T_pM$ are the principal vectors at a point $p \in M$, with respective curvatures k_1 and k_2, and if $T_pM \ni \mathbf{u} = \cos\theta \cdot \mathbf{u}_1 + \sin\theta \cdot \mathbf{u}_2$, then:

$$k(u) = \cos^2\theta \cdot k_1 + \sin^2\theta \cdot k_2$$

(1): We claim the the mean curvature H at p is the average normal curvature at p, i.e. that $H = \frac{1}{2\pi}\int_0^{2\pi} k(\theta)\ d\theta$. This claim follows from the computation:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} k(\theta)\ d\theta &= \frac{1}{2\pi}\int_0^{2\pi} k_1\cos^2\theta + k_2\sin^2\theta\ d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} k_1(1-\sin^2\theta) + k_2\sin^2\theta\ d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} k_1 + \sin^2\theta(k_2 - k_1)\ d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} k_1 + \frac{1}{2}\left(1-\cos 2\theta\right)\left(k_2 - k_1\right)\ d\theta \\
&= \frac{1}{2\pi}\left[k_1\theta + \frac{k_2-k_1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi} \\
&= \frac{1}{2\pi}\left[2\pi k_1 + \frac{k_2-k_1}{2}\cdot 2\pi\right] \\
&= k_1 + \frac{k_2-k_1}{2} \\
&= \frac{k_2+k_1}{2} \\
&= H
\end{aligned}$$

(2): Suppose $T_pM \ni \mathbf{v}_1, \mathbf{v}_2$ with $\mathbf{v}_1 \perp \mathbf{v}_2$. We claim that $H = \frac{1}{2}\left(k(\mathbf{v}_1) + k(\mathbf{v}_2)\right)$. To see this, let ϕ be such that $\mathbf{v}_1 = \cos\phi \cdot \mathbf{u}_1 + \sin\phi \cdot \mathbf{u}_2$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the principal directions as above, again with respective curvatures k_1 and k_2. Then we may assume that $\mathbf{v}_2 = \cos\left(\phi + \frac{\pi}{2}\right)\mathbf{u}_1 + \sin\left(\phi + \frac{\pi}{2}\right)\mathbf{u}_2$. (The other possibility would be with $\phi - \frac{\pi}{2}$, but then we could switch the indices of $\mathbf{v}_1$ and $\mathbf{v}_2$ and choose a new ϕ for our relabeled $\mathbf{v}_1$, giving $\mathbf{v}_2$ as above.) Now by Euler's formula, we have:

$$\begin{aligned}
\frac{k(\mathbf{v}_1) + k(\mathbf{v}_2)}{2} &= \frac{1}{2}\left[\left(\cos^2\phi + \cos^2\left(\phi + \frac{\pi}{2}\right)\right)k_1 + \left(\sin^2\phi + \sin^2\left(\phi + \frac{\pi}{2}\right)\right)k_2\right] \\
&= \frac{1}{2}\left[\left(\cos^2\phi + (-\sin\phi)^2\right)k_1 + \left(\sin^2\phi + \cos^2(\phi)\right)k_2\right] \\
&= \frac{1}{2}\left[k_1 + k_2\right] \\
&= H
\end{aligned}$$

Exercise 3.1.7: We know that K and H are given in terms of the principal curvatures $k_1 \geq k_2$ by:

$$K = k_1 k_2 \qquad \text{and} \qquad H = \frac{k_1 + k_2}{2}$$

We claim inversely that the principal curvatures are given by:

$$k_1 = H + \sqrt{H^2 - K} \qquad \text{and} \qquad k_2 = H - \sqrt{H^2 - K}$$

This follows from the computations:

$$\begin{aligned}
H + \sqrt{H^2 - K} &= \frac{k_1 + k_2}{2} + \sqrt{\frac{k_1^2 + 2k_1k_2 + k_2^2}{4} - k_1k_2} \\
&= \frac{k_1 + k_2}{2} + \frac{1}{2}\sqrt{k_1^2 - 2k_1k_2 + k_2^2} \\
&= \frac{1}{2}\left(k_1 + k_2 + \sqrt{(k_1 - k_2)^2}\right) \qquad \text{and now because } k_1 \geq k_2 : \\
&= \frac{1}{2}\left(k_1 + k_2 + (k_1 - k_2)\right) \\
&= k_1
\end{aligned}$$

and similarly

$$\begin{aligned}
H - \sqrt{H^2 - K} &= \frac{k_1 + k_2}{2} - \sqrt{\frac{k_1^2 + 2k_1k_2 + k_2^2}{4} - k_1k_2} \\
&= \frac{k_1 + k_2}{2} - \frac{1}{2}\sqrt{k_1^2 - 2k_1k_2 + k_2^2} \\
&= \frac{1}{2}\left(k_1 + k_2 - \sqrt{(k_1 - k_2)^2}\right) \\
&= \frac{1}{2}\left(k_1 + k_2 - (k_1 - k_2)\right) \\
&= k_2
\end{aligned}$$

Exercise 3.1.9: Recall from Exercise 2.2.14 that for a cylinder $\phi(u,v) = (R\cos u, R\sin u, v)$, the shape operator satisfies:

$$S(\phi_u) = -\frac{1}{R}\phi_u \qquad \text{and} \qquad S(\phi_v) = 0$$

and is therefore given with respect to the ϕ_u, ϕ_v basis by

$$S = \begin{bmatrix} -1/R & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, we have principal curvatures $k_1 = 0$ and $k_2 = -\frac{1}{R}$ with Gauss and mean curvatures given respectively by:

$$K = k_1 \cdot k_2 = 0 \qquad \text{and} \qquad H = \frac{k_1 + k_2}{2} = \frac{1}{-2R}$$

The fact that $K = 0$ means that the cylinder is flat, while the fact that $H \neq 0$ means that the cylinder is not minimal.

Exercise 3.1.10: Suppose that M is minimal, i.e. that at each point $p \in M$ the principal curvatures k_1, k_2 satisfy $0 = H = \frac{1}{2}(k_1 + k_2)$, hence $k_1 = -k_2$. It immediately follows that at each point we have:

$$K = k_1 \cdot k_2 = (-k_2) \cdot k_2 = -k_2^2 \leq 0$$

Thus, a minimal surface must everywhere have nonpositive Gaussian curvature.

Exercise 3.1.11: We claim that the sphere of radius R has Gaussian curvature $K \equiv \frac{1}{R^2}$. We will show this two ways.

First, recall from Exercise 2.2.13 that the shape operator satisfies $S(\phi_u) = -\frac{1}{R} \cdot \phi_u$ and $S(\phi_v) = -\frac{1}{R} \cdot \phi_v$, and is therefore given with respect to this basis by:

$$S = \begin{bmatrix} -1/R & 0 \\ 0 & -1/R \end{bmatrix}$$

Hence, K equals the determinant of this matrix: $K = \frac{1}{R^2}$.

Alternatively, consider that the Gauss map $G : \underbrace{M}_{=S^2_R} \to S^2$ from the sphere of radius R to the unit sphere is given by $G : \mathbf{p} \mapsto \mathbf{p}/R$. That is, G is just radial projection of the R-sphere to the unit sphere, sending any region of the R-sphere to the corresponding region of the unit sphere. Note in particular that for any region U on the R-sphere, the percentage of the area of the R-sphere which U occupies will equal the percentage of area that its image occupies on the unit sphere. Since the area of the R-sphere is $4\pi R^2$ while the area of the unit sphere is 4π, it follows that the Gauss map will send an arbitrary region U of area A on the R-sphere to a region $G(U)$ of area $\frac{A}{R^2}$ on the unit sphere. Hence:

$$|K| = \frac{\text{Area}(G(U))}{\text{Area}(U)} = \frac{1}{R^2}$$