Introduction to Differential Geometry I
Homework 8

MATH:4500

November 14, 2017

Problem 1. 2.4.4

Solution:

I. $\Leftarrow$

Suppose α is a line of curvature. For each $t \in I$, α' is an eigenvector of $S_{\alpha(t)}$. So there exists $c \in \mathbb{R}$ such that

$$S_{\alpha(t)}(\alpha'(t)) = c \cdot \alpha'(t)$$

But,$\nabla_{\alpha(t)} U(t) = -S_{\alpha(t)}(\alpha'(t)) = -c \cdot \alpha'(t)$ and thus the vectors are parallel along α.

II. $\Rightarrow$

Reverse the argument.

(b)Further, suppose α is a plane curve. Show α is a line of curvature if the angle between M and P is constant along α.

Let V be the unit normal vector field to P along α. Let U be the unit normal vector field to M along α. Notice that $V' = 0$ since V is parallel. Since the angle between U and V is constant, $\cdot V$ is constant. Thus we have,

$$0 = (U \cdot V)' = U' \cdot V + U \cdot V' = U' \cdot V.$$

Therefore, we have that U' is orthogonal to V and U (since $U \cdot U = 1$). We also have that α' is orthogonal to V and U because $\alpha' \in T_pM$ for all p along α. Because V and U are linearly independent, we have that U' and α' are colinear. Thus α' is a line of curvature. □

Problem 2. 2.4.6

Solution:

Since $\Psi(u, v) = (u, v, u^2 - v^2)$, then $\Psi_u = (1, 0, 2u)$, $\Psi_v = (0, 1, -2v)$, $\Psi_{uu} = (0, 0, 2)$, $\Psi_{uv} = (0, 0, 2)$, $\Psi_{vv} = (0, 0, -2)$.

Hence at $(0,0)$, we have $\Psi_u = (1,0,0)$, $\Psi_u = (1,0,0)$,

Therefore at $(0,0)$, we have $E = 1$, $F = 0$, $G = 1$, and $l = 2$, $m = 0$, $n = -2$.

And

$$[S_p] = [I_p]^{-1}[II_p] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

with $\lambda_1 = 2$, $\lambda_2 = -2$.

And the normal vector is $\Psi_u \times \Psi_v = (0,1,0) \times (0,0,1) = (0,0,1)$,

And $\sigma(t) = (0, t, -t^2)$, so $\sigma'(t) = (0,1,-2t)$, $\sigma''(t) = (0,0,-2)$, and we get a negative sign at here.

Therefore,

$$k(u) = k(0,1,0) = -\kappa_\sigma(0) = -\frac{|\sigma'(0) \times \sigma''(0)|}{|\sigma(0)|^3} = -2/1 = -2.$$

□

Problem 3. 2.4.7

Solution:

Since $\Psi(u,v) = (u, v, u^2 - v^2)$, then $\Psi_u = (1,0,2u)$, $\Psi_v = (0,1,-2v)$, $\Psi_{uu} = (0,0,2)$, $\Psi_{uv} = (0,0,2)$, $\Psi_{vv} = (0,0,-2)$.

Hence at $(0,0,0)$, we have $\Psi_u = (1,0,0)$, $\Psi_u = (1,0,0)$.

Therefore at $(0,0)$, we have $l = 2$, $m = 0$, $n = -2$. Since

$$[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0,$$

And this is a line with $\kappa_\sigma = 0$, and since $\overrightarrow{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{1}{\sqrt{2}}(1,0,0) + \frac{1}{\sqrt{2}}(0,1,0)$.

Thus $\sigma(t) = \left(\frac{1}{\sqrt{2}}t, \frac{1}{\sqrt{2}}t, 0\right)$ as a line.

So $\sigma'(t) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$, $\sigma''(t) = (0,0,0)$. Hence $k(u) = \kappa_\sigma(0) = 0$.

□

Problem 4. 2.4.9

Solution:

Let $M : x^2 + y^2 = 1$, $p = (1,0,0)$ and $\overrightarrow{u} = (a,b,c)$. Let $F(x,y,z) = x^2 + y^2$, $M = F^{-1}(1)$, then $\nabla F = (2x, 2y, 0)$ and $|\nabla F| = 2\sqrt{x^2+y^2}$. Also $\nabla F \perp M$.

So $U = \dfrac{\nabla F}{|\nabla F|} = \dfrac{1}{2\sqrt{x^2+y^2}}(2x, 2y, 0)$. So $U(p) = (1,0,0)$.

By $\overrightarrow{u} \in T_p(M)$, and $u \cdot (1,0,0) = 0 = (a,b,c)\cdot(1,0,0) = 0 \Longrightarrow a = 0$. and $(0,b,c)$ is a unit vector, so $b^2 + c^2 = 1$.

Additionally,

$$U \times u = \begin{bmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & b & c \end{bmatrix} = (0, -c, b),$$

which is a normal of the plane containing $\overrightarrow{u}$ and U.

Hence $-cy + bz = 0$, so $z = \dfrac{c}{b}y$.

Therefore, we can parametrize $\sigma(t) = (\sqrt{1-t^2}, t, c/b(t))$,

with $\sigma'(t) = (-t/\sqrt{1-t^2}, 1, c/b)$, $\sigma''(t) = (-1/(1-t^2)^{3/2}, 0, 0)$.

Then

$$V = |\sigma'(t)| = \sqrt{t^2/(1-t^2) + 1 + (c/b)^2}.$$

And $T = \dfrac{\sigma'(t)}{V} = \dfrac{1}{\sqrt{t^2/(1-t^2)+1+(c/b)^2}}(-t/\sqrt{1-t^2}, 1, c/b)$, thus at $t = 0$ by using $b^2 + c^2 = 1$:

$$T(0) = \frac{1}{\sqrt{1+(c/b)^2}} \cdot (0, 1, c/b) = \frac{b}{\sqrt{b^2+c^2}}(0,1,c/b) = (0,b,c).$$

And $B = \sigma'(t) \times \sigma''(t)$, thus

$$B(0) = \frac{\sigma'(0)\times\sigma''(0)}{|\sigma'(0)\times\sigma''(0)|} = \frac{1}{\sqrt{1+(c/b)^2}}(0, -c/b, 1) = (0, -c, b).$$

Therefore

$$N(0) = B(0) \times T(0) = (0,-c,b)\times(0,b,c) = (-1,0,0).$$

And this implies $N(0) = -U(p)$, so

$$k(u) = -\kappa_\sigma(0) = -\frac{|\sigma'(0)\times\sigma''(0)|}{|\sigma'(0)|^3} = -\frac{1}{1+(c/b)^2} = -\frac{1}{1+(c/b)^2}\cdot\frac{b^2}{b^2} = -b^2$$

And since $u = (0,b,c)$ is on the unit circle in the $yz-$plane, $\max k(u) = 0$ occurs when $b = 0$ and $\min k(u) = -1$ occurs when $b = \pm 1$. □