PROBLEM SET 7: §2.3

Exercise 2.3.2: Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ associated with the matrix $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$. Example 2.3.1 in the textbook shows that one eigenvector for this transformation is $\begin{bmatrix} a \\ \frac{3a}{2} \end{bmatrix}$. We claim that the other eigenvector for this transformation is $\begin{bmatrix} a \\ a \end{bmatrix}$. This follows from the computation:

$$\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 4a - 2a \\ 3a - a \end{bmatrix} = \begin{bmatrix} 2a \\ 2a \end{bmatrix} = 2 \begin{bmatrix} a \\ a \end{bmatrix}$$

Exercise 2.3.3: Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Because these are the eigenvectors for the transformation T from the previous exercise, and have respective eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, we know that the matrix for T relative to this basis is:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} =: B$$

Finally, observe that $\lambda_1 \cdot \lambda_2 = 2 = \det(B)$ and $\lambda_1 + \lambda_2 = 3 = \mathrm{tr}(B)$

Exercise 2.3.10: Consider the cylinder parameterized by $\phi(u, v) = (R \cos u, R \sin u, v)$, where $R > 0$. First, we compute the Gauss map and its derivative:

$$\begin{aligned} \phi_u &= (-R \sin u, R \cos u, 0) \\ \phi_v &= (0, 0, 1) \\ \phi_u \times \phi_v &= (R \cos u, R \sin u, 0) \\ |\phi_u \times \phi_v| &= R \\ U(u, v) &= \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = (\cos u, \sin u, 0) \\ \frac{\partial U}{\partial u} &= (-\sin u, \cos u, 0) \\ \frac{\partial U}{\partial v} &= (0, 0, 0) \end{aligned}$$

Finally, we observe that the image of the Gauss map on the sphere is just its equator $\{(x, y, z) \in \mathbb{S}^2 \mid z = 0\}$, with an area of zero.

Exercise 2.3.11: Consider the catenoid parameterized by $\phi(u,v) = (u, \cosh u \cos v, \cosh u \sin v)$, with $-\infty < u < \infty$ and $0 < v < 2\pi$. First, we compute the Gauss map and its derivative.

$$\phi_u = (1, \sinh u \cos v, \sinh u \sin v)$$
$$\phi_v = (0, -\cosh u \sin v, \cosh u \cos v)$$
$$\phi_u \times \phi_v = (\sinh u \cosh u, -\cosh u \cos v, -\cosh u \sin v)$$
$$|\phi_u \times \phi_v| = \sqrt{\cosh^2 u \left(\sinh^2 u + \cos^2 v + \sin^2 v\right)} = \cosh^2 u$$
$$U(u,v) = (\tanh u, -\text{sech}\, u \cos v, \text{sech}\, u \sin v)$$
$$\frac{\partial U}{\partial u} = (\text{sech}^2 u, -\text{sech}\, u \tanh u \cos v, \text{sech}\, u \tanh u \sin v) = \text{sech}^2 u \cdot \phi_u$$
$$\frac{\partial U}{\partial v} = (0, \text{sech}\, u \sin v, \text{sech}\, u \cos v) = -\text{sech}^2 u \cdot \phi_v$$

Finally, we argue that the map is a one-to-one map from the catenoid to the sphere. To begin, we consider the first coordinate of $U(u,v) =: (x(u,v), y(u,v), z(u,v))$, namely $x(u,v) = \tanh u = \frac{1-e^{-2x}}{1+e^{-2x}}$. Everywhere other than at the poles $(1,0,0), (-1,0,0) \in \mathbb{S}^2$ (which are not in the image of U), we have an inverse map:

$$u = \text{arctanh}\ (x(u,v)) = \frac{1}{2} \ln \left| \frac{1 + x(u,v)}{1 - x(u,v)} \right|$$

Now observe that if $U(u,v) = U(\hat{u}, \hat{v})$, then we must have $u = \hat{u}$, hence $(\cos v, \sin v) = (\cos \hat{v}, \sin \hat{v})$, with the fact that $0 < v < 2\pi$ implying that $v = \hat{v}$ as well. We conclude as claimed that U maps the catenoid injectively to $\mathbb{S}^2$.