PROBLEM SET 6: §2.2

Exercise 2.2.5 Let $M \subset \mathbb{R}^3$ be a regular surface, $p \in M$, $v \in T_pM$, and $f, g : \mathbb{R}^3 \xrightarrow{C^1} \mathbb{R}$. Then for any $\alpha : I \to M$ with $\alpha(0) = p$ and $\alpha'(0) = v$ we have:

$$\begin{aligned}
v[f \cdot g] &= \frac{d}{dt}\left((f \cdot g)(\alpha(t))\right)\Big|_{t=0} \\
&= \left(fg' + f'g\right)(\alpha(t)) \cdot \alpha'(t)\Big|_{t=0} \\
&= \left(fg'\right)(\alpha(t)) \cdot \alpha'(t)\Big|_{t=0} + \left(f'g\right)(\alpha(t)) \cdot \alpha'(t)\Big|_{t=0} \\
&= \underbrace{f(\alpha(t))\Big|_{t=0}}_{f(p)} \cdot \underbrace{g'(\alpha(t)) \cdot \alpha'(t)\Big|_{t=0}}_{v[g]} + \underbrace{g(\alpha(t))\Big|_{t=0}}_{g(p)} \cdot \underbrace{f'(\alpha(t)) \cdot \alpha'(t)\Big|_{t=0}}_{v[f]} \\
&= f(p) \cdot v[g] + g(p) \cdot v[f]
\end{aligned}$$

Exercise 2.2.8 Let $M = \{(x, y, z) \mid g(x, y, z) = c\}$ for some C^1 function $g : \mathbb{R}^3 \to \mathbb{R}$ and some $c \in \mathbb{R}$. We claim that for $\nabla g(p)$ is a normal vector for M at every $p \in M$. To see this, fix some $p \in M$, choose some arbitrary $v \in T_pM$, and let $\alpha : I \to M$ satisfy $\alpha(0) = p$ and $\alpha'(0) = v$. This allows us to confirm that $\nabla g(p) \perp v$:

$$\nabla g(p) \bullet v = \frac{d}{dt}\Big(\underbrace{g(\overbrace{\alpha(t)}^{\in M})}_{=c}\Big) = 0$$

Exercise 2.2.14 Let $R > 0$, and let $M = \left\{(x, y, z) \mid x^2 + y^2 = R^2\right\}$ be parameterized by $\phi(u, v) = (R\cos u, R\sin u, v)$. Then:

$$\phi_u = (-R\sin u, R\cos u, 0) \qquad \text{and} \qquad \phi_v = (0, 0, 1)$$

$$\phi_u \times \phi_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin u & R\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos u, R\sin u, 0)$$

$$U = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(R\cos u, R\sin u, 0)}{R} = (\cos u, \sin u, 0)$$

$$\begin{aligned}
S(\phi_u) &= -\nabla_{\phi_u} U = -\nabla_{\phi_u}(\cos u, \sin u, 0) \\
&= (\sin u, -\cos u, 0) \\
&= -\frac{1}{R}\phi_u
\end{aligned}$$

$$S(\phi_v) = -\nabla_{\phi_v} U = -\nabla_{\phi_v} (\cos u, \sin u, 0) = \mathbf{0}$$

In the u-direction, the cylinder is circular, resembling a sphere, while in the v-direction the cylinder is linear, resembling a plane.

Exercise 2.2.16 Let $M = \{ (x, y, z) \mid xy = z\}$ be parameterized by $\phi(u, v) = (u, v, uv)$. Then:

$$\phi_u = (1, 0, v) \qquad \text{and} \qquad \phi_v = (0, 1, u)$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1)$$

$$U = \frac{\phi_u \times \phi_v}{|\phi_u \times \phi_v|} = \frac{(-v, -u, 1)}{\sqrt{u^2 + v^2 + 1}}$$

$$\begin{aligned} S(\phi_u) &= -\nabla_{\phi_u} U = -\nabla_{\phi_u} \left(\frac{1}{\sqrt{u^2+v^2+1}} \cdot (-v, -u, 1) \right) \\ &= -\left(-\frac{1}{2} \left(u^2+v^2+1\right)^{-3/2} \cdot 2u \cdot (-v, -u, 1) + \left(u^2+v^2+1\right)^{-1/2} \cdot (0, -1, 0) \right) \\ &= \left(u^2+v^2+1\right)^{-3/2} \left((-uv, -u^2, u) + (0, u^2+v^2+1, 0)\right) \\ &= \left(u^2+v^2+1\right)^{-3/2} \left(-uv, 1+v^2, u\right) \\ &= \left(u^2+v^2+1\right)^{-3/2} \left(-uv, 1+v^2, -uv^2+u+uv^2\right) \\ &= \frac{-uv}{\left(u^2+v^2+1\right)^{3/2}} \phi_u + \frac{1+v^2}{\left(1+u^2+v^2\right)^{3/2}} \phi_v \end{aligned}$$

$$\begin{aligned} S(\phi_v) &= -\nabla_{\phi_v} U = -\nabla_{\phi_v} \left(\frac{1}{\sqrt{u^2+v^2+1}} \cdot (-v, -u, 1) \right) \\ &= -\left(-\frac{1}{2} \left(u^2+v^2+1\right)^{-3/2} \cdot 2v \cdot (-v, -u, 1) + \left(u^2+v^2+1\right)^{-1/2} \cdot (-1, 0, 0) \right) \\ &= \left(u^2+v^2+1\right)^{-3/2} \left((-v^2, -uv, v) + (u^2+v^2+1, 0, 0)\right) \\ &= \left(u^2+v^2+1\right)^{-3/2} \left(1+u^2, -uv, v\right) \\ &= \left(u^2+v^2+1\right)^{-3/2} \left(1+u^2, -uv, -u^2v+v+u^2v\right) \\ &= \frac{1+u^2}{\left(1+u^2+v^2\right)^{3/2}} \phi_u - \frac{uv}{\left(u^2+v^2+1\right)^{3/2}} \phi_v \end{aligned}$$