PROBLEM SET 5: §2.1

Exercise 6: Let $\phi(u,v) = (u, v, u^2 + v^2)$ be a Monge patch on the paraboloid. Then the parameter curves for $u_0 = 0$ and for $v_0 = 0$ respectively are the parabolas:

$$\phi(0,v) = (0, v, v^2) \qquad \text{and} \qquad \phi(u,0) = (u, 0, u^2)$$

Exercise 12: Consider the catenoid obtained by revolving the catenary $y = \cosh x$ about the x-axis. The catenoid can be described implicitly as $\{(x,y,z) \mid y^2 + z^2 = \cosh^2 x\}$, or equivalently $\{(x,y,z) \mid \sqrt{y^2+z^2} = \cosh x\}$. (Note that $\cosh x > 0$ for all x.) The following is a patch on the catenoid:

$$\phi(u,v) = (u, \cos v \cosh u, \sin v \cosh x) \qquad \text{with} \qquad -\infty \leq u \leq \infty \qquad \text{and} \qquad 0 \leq v \leq 2\pi$$

Exercise 19: Consider the standard cone $\left\{(x,y,z) \mid z = \sqrt{x^2+y^2}\right\}$ and the standard cylinder $\left\{(x,y,z) \mid x^2 + y^2 = 1\right\}$. These have respective ruled parameterizations:

$$\phi(u,v) = (0,0,0) + v(\cos u, \sin u, 1) \qquad \text{with} \qquad 0 \leq u \leq 2\pi \qquad \text{and} \qquad 0 \leq v < \infty$$

and

$$\psi(u,v) = (\cos u, \sin u, 0) + v(0,0,1) \qquad \text{with} \qquad 0 < u < 2\pi \qquad \text{and} \qquad -\infty < v < \infty$$

Exercise 20: Consider the saddle surface given by $z = xy = r^2 \sin\theta\cos\theta = \frac{1}{2}r^2\cos 2\theta$. We can rule the surface by:

$$\phi(u,v) = \beta(u) + v\delta(u) \qquad \text{with} \qquad \beta(u) = (0,u,0) \qquad \text{and} \qquad \delta(u) = (1,0,u)$$
$$\text{with} \qquad -\infty < u < \infty \qquad \text{and} \qquad -\infty < v < \infty$$

Exercise 24: Suppose M is a surface ruled by the parameterization $\phi(u,v) = \beta(u) + v\delta(u)$ with $|\beta'| \equiv 1$ and $|\delta| \equiv 1$. Also assume δ' is nonvanishing. We claim that M may be reparameterized by $\psi(u,w) = \gamma(u) + w\delta(u)$, where $\gamma' \bullet \delta' = 0$.

Because β and γ rule the surface, we may write $\gamma(u) = \beta(u) + r(u)\delta(u)$. Noting that $1 = \delta \bullet \delta$ implies that $0 = (\delta \bullet \delta)' = 2\delta \bullet \delta'$, (and recalling that $\delta' \bullet \delta' \neq 0$) we now have:

$$\begin{aligned}
\gamma' &= \beta' + r'\delta + r\delta' \\
0 = \gamma' \bullet \delta' &= \beta' \bullet \delta' + r' \underbrace{\delta \bullet \delta'}_{=0} + r\delta' \bullet \delta' \\
-\beta' \bullet \delta' &= r\delta' \bullet \delta' \\
-\frac{\beta'(u) \bullet \delta'(u)}{\delta'(u) \bullet \delta'(u)} &= r(u)
\end{aligned}$$

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Now letting $w = v - r(u)$, the fact that $\beta + v\delta$ rules M implies that $\gamma + w\delta$ does as well:

$$\gamma(u) + w\delta(u) = (\beta(u) + r(u)\delta(u)) + w\delta(u) = (\beta(u) + (v - w)\delta(u)) + w\delta(u) = \beta(u) + v\delta(u)$$

Such a curve γ is called a line of striction for M. Finally, we also claim that any point on M where $\psi_u \times \psi_w = (0, 0, 0)$ must lie on the line of striction. First we compute:

$$\psi_u = \gamma' + w\delta' \qquad \text{and} \qquad \psi_w = \delta$$

$$\begin{aligned} (0,0,0) &= \psi_u \times \psi_w \\ &= (\gamma' + w\delta') \times (\delta) \\ &= \gamma' \times \delta + w(\delta' \times \delta) \qquad\qquad (*) \end{aligned}$$

Now, recalling that $\delta' \perp \delta$ and $\gamma' \perp \delta$, we observe that we everywhere we have one of $\delta \times \gamma' = 0$ or $\delta' \parallel \delta \times \gamma'$. Hence, taking norm squared on both sides of $(*)$, we have:

$$0 = |\gamma' \times \delta|^2 + w^2 |\delta' \times \delta|^2 + 2w \underbrace{(\gamma' \times \delta) \bullet (\delta' \times \delta)}_{=0}$$

This can only be the case if $w = 0$, i.e. if we are on the line of striction. We conclude as claimed that the only points where $\psi_u \times \psi_w = 0$ are on the line of striction.