PROBLEM SET 4: §1.4

Exercise 6: Let $\alpha(t) = (a\cos t, b\sin t)$. Then $\alpha'(t) = (-a\sin t, b\cos t)$ and $\alpha''(t) = (-a\cos t, -b\sin t)$. Thus, $|\alpha'| = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = |\alpha''|$ (note in particular that α' is non-vanishing, i.e. α is regular) and:

$$\alpha' \times \alpha'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & b\cos t & 0 \\ -a\cos t & -b\sin t & 0 \end{vmatrix} = \mathbf{k}\cdot ab(\sin^2 t + \cos^2 t) = (0,0,ab)$$

Applying the formula $\kappa = \frac{|\alpha'\times\alpha''|}{|\alpha'|^3}$, we conclude:

$$\kappa = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}$$

Note that when α defines a circle, i.e. when $a = b$, this simplifies to:

$$\kappa = \frac{a^2}{\left(a^2\left(\sin^2 t + \cos^2 t\right)\right)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}$$

Finally, suppose $\alpha(t) = (x(t), y(t))$ is a (REGULAR) plane curve. Then $\alpha'(t) = (x'(t), y'(t))$ and $\alpha''(t) = (x''(t), y''(t))$, hence $|\alpha'| = \sqrt{(x'(t))^2 + (y'(t))^2}$, $|\alpha''| = \sqrt{(x''(t))^2 + (y''(t))^2}$ and:

$$\alpha' \times \alpha'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = \mathbf{k}\cdot x'y'' - x''y'$$

Applying the formula $\kappa = \frac{|\alpha'\times\alpha''|}{|\alpha'|^3}$, we conclude:

$$\kappa = \frac{|x'y'' - x''y'|}{\left(x'^2 + y'^2\right)^{3/2}}$$

Exercise 8: Consider the hyperbolic helix $\alpha(t) = (\cosh t, \sinh t, t)$. We have $\alpha'(t) = (\sinh t, \cosh t, 1)$, $\alpha''(t) = (\cosh t, \sinh t, 0)$, and $\alpha'''(t) = (\sinh t, \cosh t, 0)$. Thus, $\nu = |\alpha'| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cdot|\cosh t| = \sqrt{2}\cosh t$ (since $\cosh t > 0$ for all $t \in \mathbb{R}$) and:

$$T = \frac{\alpha'}{\nu} = \frac{(\sinh t, \cosh t, 1)}{\sqrt{2}\cosh t} = \boxed{\frac{1}{\sqrt{2}}(\tanh t, 1, \operatorname{sech} t) = T}$$

Also:

$$\alpha' \times \alpha'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh t & \cosh t & 1 \\ \cosh t & \sinh t & 0 \end{vmatrix} = \left(-\sinh t, \cosh t, \sinh^2 t - \cosh^2 t\right) = (-\sinh t, \cosh t, -1)$$

Therefore:

$$B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} = \frac{(-\sinh t, \cosh t, -1)}{\sqrt{\sinh^2 t + \cosh^2 t + 1}} = \frac{(-\sinh t, \cosh t, -1)}{\sqrt{2\cosh^2 t}}$$

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$$= \frac{1}{\sqrt{2}\cosh t} \cdot (-\sinh t, \cosh t, -1) = \boxed{\frac{1}{\sqrt{2}}(-\tanh t, 1, -\text{sech } t) = B}$$

Continuing, we have:

$$N = B \times T = \frac{\sqrt{2}T \times \sqrt{2}B}{2} = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\tanh t & 1 & -\text{sech } t \\ \tanh t & 1 & \text{sech } t \end{vmatrix} = \frac{1}{2}(2 \text{ sech } t, 0, -2\tanh t) = \boxed{(-\text{ sech } t, 0, \tanh t) = N}$$

$$\kappa = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} = \frac{\sqrt{2}\cosh t}{\left(\sqrt{2}\cosh t\right)^3} = \frac{1}{2\cosh^2 t} = \kappa$$

$$\tau = \frac{(\alpha' \times \alpha'') \bullet \alpha'''}{|\alpha' \times \alpha''|^2} = \frac{(-\sinh t, \cosh t, -1) \bullet (\sinh t, \cosh t, 0)}{\left(\sqrt{2}\cosh t\right)^2} = \frac{-\sinh^2 t + \cosh^2 t + 0}{2\cosh^2 t} = \frac{1}{2\cosh^2 t} = \tau$$

Exercise 9: Let $\alpha(t)$ be the hyperbolic helix from the previous exercise. We want to compute $\int_0^t |\alpha'(u)| \; du$. Having already computed $|\alpha'(t)| = \sqrt{2}\cosh t$, we simply have:

$$\int_0^t |\alpha'(u)| \; du = \sqrt{2}\int_0^t \cosh u \; du = \sqrt{2}\Big[\sinh t\Big]_0^t = \sqrt{2}\,(\sinh t - \sinh 0) = \boxed{\sqrt{2}\sinh t}$$

Exercise 10: Let $\alpha(t)$ be a (not necessarily unit speed) curve with involute $\mathcal{I}(t) = \alpha(t) - s(t)\frac{\alpha'(t)}{|\alpha'(t)|} = \alpha(t) - s(t) \cdot T_\alpha(t)$. We claim that the curvature of the involute $\mathcal{I}(t)$ is given by:

$$\kappa_{\mathcal{I}}(t) = \frac{\sqrt{\kappa_\alpha(t)^2 + \tau_\alpha(t)^2}}{s(t) \cdot \kappa_\alpha(t)}$$

where $s(t)$ is the arclength function for α. To simplify the notation as we justify this claim, we will omit the t's. Also, we will assume that $\kappa_\alpha(t)$ is nonvanishing, so that the RHS of the formula is defined.

Using the product rule to differentiate our formula for $\mathcal{I}$, we obtain:

$$\begin{aligned} \mathcal{I}' &= \alpha' - s' \cdot T_\alpha - s \cdot T_\alpha' = \nu_\alpha T_\alpha - \nu_\alpha T_\alpha - s \cdot \kappa_\alpha \nu_\alpha N_\alpha = -s \cdot \kappa_\alpha \cdot \nu_\alpha \cdot N_\alpha \\ \mathcal{I}'' &= -s' \cdot \kappa_\alpha \cdot \nu_\alpha \cdot N_\alpha - s \cdot \kappa_\alpha' \cdot \nu_\alpha \cdot N_\alpha - s \cdot \kappa_\alpha \cdot \nu_\alpha' \cdot N_\alpha - s \cdot \kappa_\alpha \cdot \nu_\alpha \cdot N_\alpha' \\ &= N_\alpha\left(-s'\kappa_\alpha\nu_\alpha - s\kappa_\alpha'\nu_\alpha - s\kappa_\alpha\nu_\alpha'\right) + N_\alpha'\left(-s \cdot \kappa_\alpha \cdot \nu_\alpha\right) \end{aligned}$$

This gives:

$$
\begin{aligned}
|\mathcal{I}'| &= s \cdot \kappa_\alpha \cdot \nu_\alpha \\
\mathcal{I}' \times \mathcal{I}'' &= \Big(N_\alpha \cdot (-s \cdot \kappa_\alpha \cdot \nu_\alpha)\Big) \times \Big(N_\alpha\,(-s'\kappa_\alpha\nu_\alpha - s\kappa'_\alpha\nu_\alpha - s\kappa_\alpha\nu'_\alpha) + N'_\alpha\,(-s \cdot \kappa_\alpha \cdot \nu_\alpha)\Big) \\
&= \underbrace{\Big(N_\alpha \cdot (-s\kappa_\alpha\nu_\alpha)\Big) \times \Big(N_\alpha\,(-s'\kappa_\alpha\nu_\alpha - s\kappa'_\alpha\nu_\alpha - s\kappa_\alpha\nu'_\alpha)\Big)}_{=0} + \Big(N_\alpha \cdot (-s \cdot \kappa_\alpha \cdot \nu_\alpha)\Big) \times \Big(N'_\alpha\,(-s \cdot \kappa_\alpha \cdot \nu_\alpha)\Big) \\
&= \Big(N_\alpha \cdot (-s \cdot \kappa_\alpha \cdot \nu_\alpha)\Big) \times \Big(N'_\alpha\,(-s \cdot \kappa_\alpha \cdot \nu_\alpha)\Big) \\
&= s^2 \cdot k_\alpha^2 \cdot \nu_\alpha^2 \cdot (N_\alpha \times N'_\alpha) \\
&= s^2 \cdot k_\alpha^2 \cdot \nu_\alpha^2 \cdot (N_\alpha \times (-\kappa_\alpha\nu_\alpha T_\alpha + \tau_\alpha\nu_\alpha B_\alpha)) \\
&= s^2 \cdot k_\alpha^2 \cdot \nu_\alpha^2 \cdot (-\kappa_\alpha\nu_\alpha B_\alpha + \tau_\alpha\nu_\alpha T_\alpha) \\
&= s^2 \cdot k_\alpha^2 \cdot \nu_\alpha^3 \cdot (-\kappa_\alpha B_\alpha + \tau_\alpha T_\alpha) \\
|\mathcal{I}' \times \mathcal{I}''| &= s^2 \cdot k_\alpha^2 \cdot \nu_\alpha^3 \cdot \sqrt{\kappa_\alpha^2 + \tau_\alpha^2}
\end{aligned}
$$

Now we conclude as claimed:

$$\kappa_{\mathcal{I}}(t) = \frac{|\mathcal{I}' \times \mathcal{I}''|}{|\mathcal{I}'|^3} = \frac{s^2 \cdot k_\alpha^2 \cdot \nu_\alpha^3 \cdot \sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}{(s \cdot \kappa_\alpha \cdot \nu_\alpha)^3} = \frac{\sqrt{\kappa_\alpha^2 + \tau_\alpha^2}}{s \cdot \kappa_\alpha}$$

If α is a plane curve, then $\tau \equiv 0$, and this becomes:

$$\kappa_{\mathcal{I}}(t) = \frac{\sqrt{\kappa_\alpha\,(t)^2}}{s(t)\kappa_\alpha(t)} = \frac{1}{s(t)}$$

(Note above that $\sqrt{\kappa_\alpha\,(t)^2} = +\kappa_\alpha\,(t)$, since everywhere we have by assumption $\kappa_\alpha > 0$.)

Finally, we compute the curvature of the involute of the unit circle directly and from this formula. Directly, we have $\alpha(t) = (\cos t, \sin t, 0)$, with $\mathcal{I}'(t) = (-\sin(t), \cos(t), 0)$, $|\mathcal{I}'(t)| = \sin^2 t + \cos^2 t = 1$, and $s(t) = \int_0^t |\mathcal{I}'(u)|\ du = t$, so that:

$$
\begin{aligned}
\mathcal{I}(t) &= \alpha(t) - s(t)\frac{\alpha'(t)}{|\alpha'(t)|} = (\cos t, \sin t, 0) - t \cdot \frac{(-\sin t, \cos t, 0)}{1} = (\cos t + t\sin t, \sin t - t\cos t, 0) \\
\mathcal{I}'(t) &= (-\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 0) = (t\cos t, t\sin t, 0) \\
\mathcal{I}''(t) &= (\cos t - t\sin t, \sin t + t\cos t, 0) \\
\mathcal{I}' \times \mathcal{I}'' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix} = \big(0, 0, t\left(\cos t\sin t + t\cos^2 t - \cos t\sin t + t\sin^2 t\right)\big) = \left(0, 0, t^2\right) \\
\kappa_{\mathcal{I}}(t) &= \frac{|\mathcal{I}' \times \mathcal{I}''|}{|\mathcal{I}'|^3} = \frac{\left|\left(0, 0, t^2\right)\right|}{|(t\cos t, t\sin t, 0)|^2} = \frac{t^2}{t^3} = \frac{1}{t}
\end{aligned}
$$

In order to apply the formula, we first note that α is a plane curve, and so having already computed $s(t) = t$, we conclude as before: $\kappa_{\mathcal{I}}(t) = 1/s(t) = 1/t$.