Introduction to Differential Geometry I
Homework 3

MATH:4500

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Problem 1. 1.3.22

Solution:

Since $\mathbf{p} = \beta(s) + r(s)\beta'(s)$, for some function $r(s)$.

Differentiate both sides, we have

$$
\begin{aligned}
0 =& \beta'(s) + r'(s)\beta'(s) + r(s)\beta''(s),\\
0 =& T(1 + r'(s)) + r(s)\beta''(s),\\
0 =& T(1 + r'(s)) + r(s)T'(s).
\end{aligned}
$$

Take the dot product with T on both side of the above equation gives us:

$$
\begin{aligned}
0 =& 1 + r'(s) + r(s)T' \cdot T,\\
0 =& 1 + r'(s).
\end{aligned}
$$

Recall that $|T| = 1 \Longrightarrow T \cdot T' = 0$.

So $r'(s) = -1 \neq 0, \Longrightarrow r(s) \not\equiv 0$.

Then take dot product with T' on both sides of $0 = T(1 + r'(s)) + r(s)T'(s)$, we have:

$$0 = 0 + r(s)|T'(s)|^2, \quad \text{i.e.,} \quad r(s)|T'(s)|^2 = 0.$$

Since we have $r(s) \not\equiv 0$, then $T'(s) = \beta''(s) = 0$, hence $\beta'(s) \equiv$ constant.

Therefore $\beta(s)$ is a line. □

Problem 2. 1.3.23

Solution:

Since $\{T, N, B\}$ an orthogonal basis, we can take $\alpha(s) - p = aT + bN + cB$ such that $T \cdot (\alpha - p) = a$, $N \cdot (\alpha - p) = b$, $B \cdot (\alpha - p) = c$.

Since α lies on a sphere pf center p and radius R, we have $(\alpha - p) \cdot (\alpha - p) = R^2$.

Take derivatives of both sides of the above equation, we have

$$2T \cdot (\alpha - p) = 0 \Longrightarrow a = T \cdot (\alpha - p) = 0. \tag{1}$$

Take derivatives again of both sides of the above equation 1, we have

$$T' \cdot (\alpha - p) + T \cdot T = 0 \Longrightarrow T' \cdot (\alpha - p) + 1 = 0.$$

And by $T' \neq 0, \kappa \neq 0$; N and B exist, we have

$$\kappa N \cdot (\alpha - p) + 1 = 0. \tag{2}$$

Take derivatives again of both sides of the above equation 2, we have

$$\begin{aligned} \kappa' N \cdot (\alpha - p) + \kappa N' \cdot (\alpha - p) + \kappa N \cdot T =& 0, \\ \Longrightarrow \kappa' N \cdot (\alpha - p) + \kappa(-\kappa T + \tau B) \cdot (\alpha - p) =& 0, \\ \Longrightarrow \kappa' N \cdot (\alpha - p) + \kappa\tau B \cdot (\alpha - p) =& 0. \end{aligned} \tag{3}$$

From 2 we have $b = N \cdot (\alpha - p) = -\frac{1}{\kappa}$, so plug into 3 then we have

$$-\frac{\kappa'}{\kappa} + \kappa\tau B \cdot (\alpha - p) = 0 \Longrightarrow B \cdot (\alpha - p) = \frac{\kappa'}{\kappa^2 \tau} = c \tag{4}$$

Therefore, since

$$\begin{aligned} \alpha(s) - p =& aT + bN + cB, \\ =& -\frac{1}{\kappa}N + \frac{\kappa'}{\kappa^2\tau}B, \\ =& -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B. \end{aligned}$$

And by $R^2 = (\alpha - p) \cdot (\alpha - p) = a^2 + b^2 + c^2$, we conclude that:

$$R^2 = \frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2.$$

□

Problem 3. 1.3.27

Solution:

Since $\beta'(s) = \left(-\frac{1}{\sqrt{2}} \sin s, \cos s, -\frac{1}{\sqrt{2}} \sin s\right)$,

then $|\beta'(s)| = \sqrt{(\cos^2 s + \sin^2 s)} = 1$, i.e., β has unit speed.

Therefore,

$$\begin{aligned} T =& \beta'(s), \\ T' =& \beta''(s) = \left(-\frac{1}{\sqrt{2}}\cos s, -\sin s, -\frac{1}{\sqrt{2}}\cos s\right), \\ \implies |T'| = \sqrt{\cos^2 s + \sin^2 s} = 1. \end{aligned}$$

Hence, $\kappa = |T'| = 1$, and $N = T' \cdot \frac{1}{\kappa} = \left(-\frac{1}{\sqrt{2}}\cos s, -\sin s, -\frac{1}{\sqrt{2}}\cos s\right)$.

So

$$B = T \times N = \begin{vmatrix} i & j & k \\ -\frac{1}{\sqrt{2}}\sin s & \cos s & -\frac{1}{\sqrt{2}}\sin s \\ -\frac{1}{\sqrt{2}}\cos s & -\sin s & -\frac{1}{\sqrt{2}}\cos s \end{vmatrix} = \left(-\frac{\cos^2 s + \sin^2 s}{\sqrt{2}}, 0, \frac{\cos^2 s + \sin^2 s}{\sqrt{2}}\right)$$

Thus $B = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $\quad \implies B' = (0,0,0) = -\tau N \implies \tau = 0.$

Therefore, we identify the curve as a circle by the fact that $\kappa = 1$ and $\tau = 0$. Indeed β lies on the intersection of the plane $x = z$ and the unit sphere. □

Problem 4. 1.3.28

Solution:

We are given a curve $\alpha(s)$, and we know $\alpha(0), \alpha'(0), \alpha''(0)$ and $\kappa(0)$. We will construct a circle $\beta(s)$ with $\beta(0) = \alpha(0), \beta'(0) = \alpha'(0), \beta''(0) = \alpha''(0)$ when $\kappa(s) > 0$.

Assume $\beta(s) = p + R\cos(\frac{s}{R})v_1 + R\sin(\frac{s}{R})v_2$ where v_1, v_2 are orthonormal vectors, and we will show how to choose p, v_1, v_2.

By $\beta(0) = \alpha(0), \beta'(0) = \alpha'(0), \beta''(0) = \alpha''(0)$, so

$$\begin{aligned} \beta(0) =& p + Rv_1 = \alpha(0), && (5) \\ \beta'(0) =& v_2 = \alpha'(0), && (6) \\ \beta''(0) =& \frac{1}{R}v_1 = \alpha''(0). && (7) \end{aligned}$$

From 7,

$$|\beta''(0)| = |\alpha''(0)| = \frac{1}{R} \implies R = \frac{1}{|\beta''(0)|} = \frac{1}{\kappa(0).} \qquad (8)$$

From 8,

$$v_1 = -R\beta''(0) = \frac{\beta''(0)}{\kappa(0)} = \frac{T'(0)}{\kappa(0)} = -N(0), \qquad (9)$$

Combine with 5, 6, we know that

$$v_2 = \beta'(0) = T(0) = \alpha'(0),$$
$$p = \beta(0) + \frac{1}{\kappa(0)} N(0).$$

Therefore, we derive a circle β as $\beta(s) = p + R\cos(\frac{s}{R})v_1 + R\sin(\frac{s}{R})v_2$, satisfying $\beta(0) = \alpha(0), \beta'(0) = \alpha'(0), \beta''(0) = \alpha''(0)$,

with the following values:

$$\begin{aligned} p =& \alpha(0) + \frac{1}{\kappa(0)} N(0), \\ v_1 =& -N(0), \\ v_2 =& T(0). \end{aligned}$$

i.e., β lies in the plane spanned by T and N.

To show the uniqueness of such circle β.

Let $\gamma(s)$ be another circle such that $\gamma(0) = \alpha(0), \gamma'(0) = \alpha'(0), \gamma''(0) = \alpha''(0)$.

From this condition, we have that γ and β share the same following quantities at $s = 0$:

- the point $\alpha(0)$,
- the tangent vector $T(0)$,
- the normal vector $N(0)$,
- the bi-normal $B(0) = T(0) \times N(0)$, which $B(0)$ is constant,
- the curvature $\kappa(0)$, which is constant as well.

then we may have that

1. γ and β lie in the same plane (which is $\perp B(0)$),
2. γ and β have the same radius $\dfrac{1}{\kappa(0)}$,
3. γ and β have the same center as $\alpha(0) + \dfrac{1}{\kappa(0)} N(0)$.

Hence, they are identical. This completes our proof for both existence and uniqueness.

When $\alpha''(0) = 0, \kappa(0) = 0$, the oscillating circle will be replaced by the tangent line:

$$\beta(s) = \alpha(0) + s\alpha'(0).$$

And we have:

$$\begin{aligned} \beta(0) =& \alpha(0) + 0 \cdot \alpha'(0) = \alpha(0), \\ \beta'(s) =& \alpha'(0) \Longrightarrow \beta'(0) = \alpha'(0), \\ \beta''(s) =& 0 \Longrightarrow \beta''(0) = \alpha''(0). \end{aligned}$$

□