Introduction to Differential Geometry I
Homework 2

MATH:4500

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Problem 1. 1.2.5

Solution:

$$\begin{aligned}
\alpha(t) =& (r\cos t, r\sin t),\\
\Longrightarrow \alpha'(t) =& (-r\sin t, r\cos t),\\
\Longrightarrow |\alpha'(t)| =& \sqrt{r^2(\sin^2 t + \cos^2 t)} = r,\\
\Longrightarrow s(t) =& \int_0^t r du = rt, t(s) = \frac{s}{r}.
\end{aligned}$$

Then, the re-parametrize of the circle with radius r is

$$\beta(s) = \alpha(t(s)) = \alpha(\frac{s}{r}) = (r\cos\frac{s}{r}, r\sin\frac{s}{r}), \quad \text{with } |\beta'(s)| = 1.$$

□

Problem 2. 1.2.7

Solution:

Show that the curve $\mathcal{I}(t) = \alpha(t) - s(t)\dfrac{\alpha'(t)}{|\alpha'(t)|}$.

Consider a given (and fixed) time t_0, the this time t_0, the swept point of $\mathcal{I}(t)$ can be described as:

$$\overrightarrow{\mathcal{I}(t_0)} = \overrightarrow{P_0} + d \cdot \overrightarrow{V_0}, \tag{1}$$

where $\overrightarrow{P_0}$ is the current starting point on $\alpha(t)$, d is the distance of the free end with $\alpha(t)$ which equals $s(t)$ as mentioned in the context, $\overrightarrow{V_0}$ is the normalized direction vector at the moment t_0, and the direction given by $-\alpha'(t)$.

Hence, by the above equation we derive a description to $\mathcal{I}(t)$.

Now, let's check these values in detail. By the above explanation:

$$\vec{P_0} = \overrightarrow{\alpha(t_0)}, \quad d = s(t_0), \quad \text{and} \quad \vec{V_0} = -\frac{\alpha'(t_0)}{|\alpha'(t_0)|}.$$

Thus (1) becomes: $\overrightarrow{\mathcal{I}(t_0)} = \overrightarrow{\alpha(t_0)} + s(t_0) \cdot \left(-\frac{\alpha'(t_0)}{|\alpha'(t_0)|}\right)$ for the moment t_0.

Therefore, expand the above reasoning for any $t \in \mathbb{R}$, we can have that:

$$\mathcal{I}(t) = \alpha(t) - s(t)\frac{\alpha'(t)}{|\alpha'(t)|},$$

as desired function.

By the given unit circle $\alpha(t)$ as following:

$$\begin{aligned} \alpha(t) =& (\cos t, \sin t, 0), \\ \Longrightarrow \alpha'(t) =& (-\sin t, \cos t, 0), \\ \Longrightarrow |\alpha'(t)| =& \sqrt{(\sin^2 t + \cos^2 t + 0)} = 1, \\ \Longrightarrow s(t) =& \int_0^t 1du = t, \\ \text{and} \quad \mathcal{I}(t) =& \alpha(t) - s(t)\frac{\alpha'(t)}{|\alpha'(t)|}, \\ \Longrightarrow \mathcal{I}(t) =& (\cos t + t\sin t, \sin t - t\cos t, 0). \end{aligned}$$

Graph of the involute of the unit circle $\alpha(t)$:

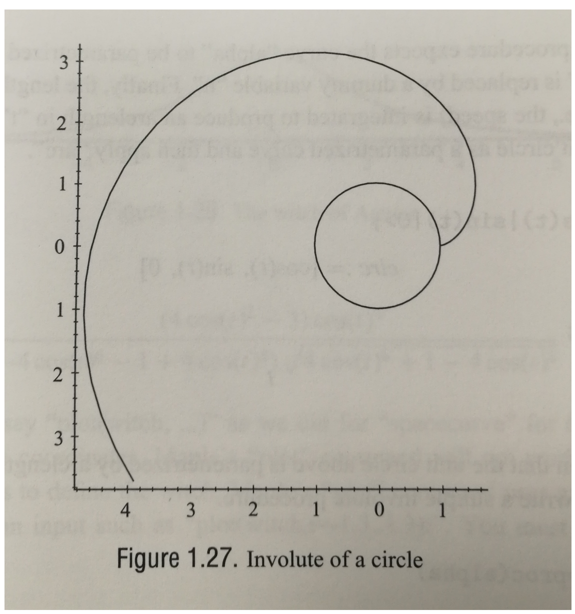

Figure 1.27. Involute of a circle

□

Problem 3. 1.2.8

Solution:

Consider the helix as following: $\alpha(t) = (R\cos\omega t, R\sin\omega t, ht)$, then:

$$\begin{aligned}
\alpha(t) =&(R\cos\omega t, R\sin\omega t, ht),\\
\implies \alpha'(t) =&(-\omega R\sin\omega t, \omega R\cos t, h),\\
\implies |\alpha'(t)| =&\sqrt{\omega^2R^2(\sin^2\omega t+\cos^2\omega t)+h^2} = \sqrt{\omega^2R^2+h^2},\\
\implies s(t) =& \int_0^t \sqrt{\omega^2R^2+h^2}du = \sqrt{\omega^2R^2+h^2}\cdot t,\\
\text{and}\quad \mathcal{I}(t) =&\alpha(t)-s(t)\frac{\alpha'(t)}{|\alpha'(t)|},
\end{aligned}$$

$$\begin{aligned}
\implies \mathcal{I}(t) &= (R\cos\omega t, R\sin\omega t, ht) - \sqrt{\omega^2R^2+h^2}\cdot t\frac{(-\omega R\sin\omega t, \omega R\cos t, h)}{\sqrt{\omega^2R^2+h^2}},\\
&= (R(\cos\omega t+\omega t\sin\omega t), R(\sin\omega t-\omega t\cos\omega t), 0).
\end{aligned}$$

Therefore, the involute of this helix lies on the xy-plane and hence it is a plane curve.

In general, every helix is of the form $\alpha(t) = (R\cos\omega t)\cdot\overrightarrow{x} + (R\sin\omega t)\cdot\overrightarrow{y} + (ht)\cdot\overrightarrow{z}$ for an orhonormal basis $\{\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}\}$ of $\mathbb{R}^3$. The involute of such a helix lies on the plane perpendicular to $\overrightarrow{z}$, and containing the starting point $\alpha(0)$ of the involute. The calculation is similar to the one done above. □

Problem 4. 1.3.11

Solution:

1. By the context, we have:

$$\begin{aligned}
\beta(s) &= \left(\tfrac{(1+s)^{3/2}}{3}, \tfrac{(1-s)^{3/2}}{3}, \tfrac{s}{\sqrt{2}}\right),\\
\implies \beta'(s) &= \left(\frac{\frac{3}{2}(1+s)^{1/2}}{3}, -\frac{\frac{3}{2}(1-s)^{1/2}}{3}, \frac{\sqrt{2}}{2}\right),\\
&= \left(\tfrac{(1+s)^{1/2}}{2}, -\tfrac{(1-s)^{1/2}}{2}, \tfrac{\sqrt{2}}{2}\right),\\
\implies |\beta'(s)| &= \sqrt{\tfrac{1+s}{4}+\tfrac{1-s}{4}+\tfrac{1}{2}} = \sqrt{1} = 1.
\end{aligned}$$

   Since $|\beta'(s)| = 1$, then β has unit speed.

2. By $T(s) = \beta'(s) = \left(\tfrac{(1+s)^{1/2}}{2}, -\tfrac{(1-s)^{1/2}}{2}, \tfrac{\sqrt{2}}{2}\right)$.

Then

$$
\begin{aligned}
T'(s) &= \left(\frac{\frac{1}{2}(1+s)^{-1/2}}{2}, \frac{\frac{1}{2}(1-s)^{-1/2}}{2}, 0\right), \\
&= \left(\frac{(1+s)^{-1/2}}{4}, \frac{(1-s)^{-1/2}}{4}, 0\right). \\
\text{And} \quad \kappa = |T'(s)| &= \sqrt{\frac{1}{16(1+s)} + \frac{1}{16(1-s)} + 0}, \\
&= \sqrt{\frac{1}{8(1-s^2)}}.
\end{aligned}
$$

3. Since $N = \dfrac{T'}{\kappa} = \left(\frac{(1+s)^{-1/2}}{4}, \frac{(1-s)^{-1/2}}{4}, 0\right) / \sqrt{\frac{1}{8(1-s^2)}} = \left(\frac{\sqrt{2(1-s)}}{2}, \frac{\sqrt{2(1+s)}}{2}, 0\right)$.

   And

$$
B = T \times N = \begin{vmatrix} i & j & k \\ \frac{(1+s)^{1/2}}{2} & -\frac{(1-s)^{1/2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2(1-s)}}{2} & \frac{\sqrt{2(1+s)}}{2} & 0 \end{vmatrix} = \left(-\frac{\sqrt{(1+s)}}{2}, \frac{\sqrt{(1-s)}}{2}, \frac{\sqrt{2}}{2}\right).
$$

4. From B above, we get $B' = \left(-\frac{1}{4\sqrt{(1+s)}}, -\frac{1}{4\sqrt{(1-s)}}, 0\right)$.

   Thus

$$
\begin{aligned}
\tau = -N \cdot B' &= -\left(\frac{\sqrt{2(1-s)}}{2}, \frac{\sqrt{2(1+s)}}{2}, 0\right) \cdot \left(-\frac{1}{4\sqrt{(1+s)}}, -\frac{1}{4\sqrt{(1-s)}}, 0\right), \\
&= \frac{\sqrt{2}}{8}\left(\frac{\sqrt{1-s}}{\sqrt{1+s}} + \frac{\sqrt{1+s}}{\sqrt{1-s}}\right), \\
&= \sqrt{\frac{1}{8(1-s^2)}}, \\
&= \kappa.
\end{aligned}
$$

$\square$