Introduction to Differential Geometry I
Homework 12

MATH:4500

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Problem 1. 5.1.1

Solution:

Let ϕ be the angle between N and U.

and $\kappa_g = \kappa_\alpha (T \times N) \cdot U = \kappa_\alpha (N \times U) \cdot T$. (Triple Product prop)

Then:

$$\begin{aligned} k(\alpha')^2 + \kappa_g^2 =& [\kappa_\alpha (N \cdot U)]^2 + [\kappa_\alpha (N \times U) \cdot T]^2 \\ =& [\kappa_\alpha \cos(\phi)]^2 + [\kappa_\alpha |N||U| \sin(\phi) \cdot T]^2 \\ =& \kappa_\alpha (\cos^2(\phi) + \sin^2(\phi)) \\ =& \kappa_\alpha^2. \end{aligned}$$

□

Problem 2. 5.1.2

Solution:

The path for torus is $X(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$, and with $u = \dfrac{\pi}{2}$, we have

$$\begin{aligned} \alpha(t) &= (R\cos t, R\sin t, r) \\ \alpha'(t) &= (-R\sin t, R\cos t, 0) \\ \alpha''(t) &= (-R\cos t, -R\sin t, 0) \\ \alpha'(t) \times \alpha''(t) &= (0, 0, R^2). \end{aligned}$$

Then

$$\kappa_\alpha = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'|^3} = \frac{1}{R}, \tag{1}$$

and $N = (-\cos t, -\sin t, 0)$.

Hence

$$\begin{aligned}\mathbf{X}_u =&(-r\sin u\cos v, -r\sin u\sin v, r\cos u),\\ \mathbf{X}_v =&(-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0),\\ \mathbf{X}_u\times\mathbf{X}_v =&-r(R+r\cos u)(\cos u\cos v, \cos u\sin v, \sin u).\end{aligned}$$

And $U = \dfrac{\mathbf{X}_u\times\mathbf{X}_v}{|\mathbf{X}_u\times\mathbf{X}_v|} = -(\cos u\cos v, \cos u\sin v, \sin u), \Longrightarrow U_\alpha = (0,0,-1)$.

Thus $U\cdot(\alpha''(t)\times\alpha'(t)) = (0,0,-1)\cdot(0,0,-R^2) = R^2$, and

$$\cos\theta = \frac{U\cdot(\alpha''(t)\times\alpha'(t))}{|U||\alpha''(t)\times\alpha'(t)|} = \frac{R^2}{R^2} = 1 \Longrightarrow \theta = 0,$$

and $\kappa_g = \kappa_\alpha\cos\theta = \kappa_\alpha = \dfrac{1}{R}$.

And by def, torus is a surface of revolution, and $\kappa_\pi = \dfrac{g'}{h\sqrt{g'^2h'^2}}$ with $g(u) = r\sin u$, $h(u) = R + r\cos u$.

So $\kappa_\pi = \dfrac{r\cos u}{-r\sin u}$, and when $u = \dfrac{\pi}{2}$, $\kappa_\pi = 0 \Longrightarrow k(\alpha') = 0$.

Thus, $\kappa_\alpha^2 = \dfrac{1}{R^2}$, $\kappa_g^2 = \dfrac{1}{R^2}$, $k(\alpha') = 0$.

And we have $k(\alpha')^2 + \kappa_g^2 = \kappa_\alpha^2$ □

Problem 3. 5.1.3

Solution:

Since $\beta(s) = \alpha(s) + \rho(s)U_0$, then $\beta'(s) = \alpha'(s) + \rho'(s)U_0$, $\beta''(s) = \alpha''(s) + \rho''(s)U_0$.

By $\rho(0) = 0, \rho'(0) = 0, \rho''(0) = -k_a$. We have,

$$\begin{aligned}\beta(0) =&0,\\ \beta'(0) =&\alpha'(0),\\ \beta''(0) =&\alpha''(0) - k_aU_0.\end{aligned}$$

So

$$\begin{aligned}\alpha'' =&(\alpha''\cdot T)T + (\alpha''\cdot U\times T)U\times T + (\alpha''\cdot U)\cdot U\\ =&0T + \kappa_g U\times T + k_\alpha\cdot U.\end{aligned}$$

by $|\alpha'(0) = 1| \Longrightarrow \alpha'\perp\alpha''$.

So

$$\alpha'' = \kappa_g(U\times T) + k_a\cdot U.$$

And

$$\beta''(s) = \alpha''(s) + \rho''(s)U_0.$$

,

then

$$\begin{aligned}\beta''(0) =& \kappa_g(0)(U \times T)(0) + k_a U_0 + \rho''(0) U_0 \\ =& \kappa_g(0)(U \times T)(0) + k_a U_0 - k_a U_0 \\ =& \kappa_g(0)(U \times T)(0).\end{aligned}$$

$$\beta'(0) = \alpha'(0) + \rho(0) U_0 = \alpha'(0) = T(0).$$

By $\beta'(0) = T(0)$, $\beta''(0) = U \times T(0)$, $|\beta'(0)| = 1$,

and $\beta'(0) \perp \beta''(0) \Longrightarrow |\beta'(0) \times \beta''(0)| = |\beta'(0)| \cdot |\beta''(0)| = 1 \cdot |\kappa_g(0)|$.

Therefore,

$$\kappa_\beta(0) = |\beta'(0) \times \beta''(0)| = |(\kappa_g)_\alpha)|.$$

□