Introduction to Differential Geometry I
Homework 11

MATH:4500

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Problem 1. 3.3.3

Solution:

Since $x(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$,

then $h(u) = R + r\cos u$, $g(u) = r\sin u$, and $h'(u) = -r\sin u$, $h''(u) = -r\cos u$, $g'(u) = r\cos u$, $g''(u) = -r\sin u$.

Thus by the formula of Gauss curvature,

$$\begin{aligned} K =& \frac{r\cos u(-r\sin u(-r\sin u) - (-r\cos u)(r\cos u))}{(R + r\cos u)(r^2\cos^2 u + r^2\sin^2 u)^2} \\ =& \frac{r\cos u(r^2)}{(R + r\cos u)(r^4)} \\ =& \frac{\cos u}{r(R + r\cos u)}. \end{aligned}$$

Moreover, since the maximum K occur at $u = 0$,

$$K_{\max} = \frac{1}{r(R + r)}.$$

And the minimum K occur at $u = \pi$,

$$K_{\min} = -\frac{1}{r(R - r)}.$$

□

Problem 2. 3.3.6

Solution:

Since $x(u,v) = (u, c\cosh(\frac{u}{c})\sin v, c\cosh(\frac{u}{c})\cos v)$,

then $h(u) = c\cosh(\frac{u}{c})$, $g(u) = u$, and $h'(u) = \sinh(\frac{u}{c})$, $h''(u) = \frac{1}{c}\cosh(\frac{u}{c})$, $g'(u) = 1$, $g''(u) = 0$.

Thus by the formula of Gauss curvature,

$$\sigma = \sqrt{\sinh^2(\frac{u}{c}) + 1} = \cosh(\frac{u}{c}).$$

and

$$k_\pi = \frac{1}{c\cosh^2(\frac{u}{c})},$$

$$\begin{aligned} k_\mu =& \frac{1}{\cosh^3(\frac{u}{c})}\left(\sinh(\frac{u}{c})(0) - \frac{1}{c}\cosh(\frac{u}{c})\right) \\ =& -\frac{1}{c\cosh^2(\frac{u}{c})} \\ =& -k_\pi. \end{aligned}$$

Thus $K = k_\pi k_\mu = -\dfrac{1}{c^2\cosh^4(\frac{u}{c})}$, and $H = k_\pi + k_\mu = 0$.

□

Problem 3. 3.4.5

Solution:

Part a:

$$\begin{aligned} & -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right) \\ =& -\frac{1}{2\sqrt{EG}}\left(\frac{E_{vv}}{\sqrt{EG}} + E_v\left(-\frac{1}{2}(EG)^{-3/2}(E_vG + EG_v)\right) + \frac{G_{uu}}{\sqrt{EG}} + G_u(-\frac{1}{2}(EG)^{-3/2}(E_uG + EG_u))\right) \\ =& -\frac{1}{2}\left(\frac{E_{vv}}{EG} - \frac{1}{2}\frac{E_v(E_vG + EG_v)}{E^2G^2} + \frac{G_{uu}}{EG} - \frac{1}{2}\frac{G_u(E_uG + EG_u)}{E^2G^2}\right). \end{aligned} \tag{1}$$

Now check RHS,

$$\text{RHS} = \frac{E_uG_uG - 2E_{vv}EG + E_vG_vE + E_vE_vG - 2G_{uu}EG + G_u^2}{4E^2G^2}. \tag{2}$$

see (1)=(2).

Part b: Since $F = 0$,

using the Prop 1 in lecture we have:

$$\begin{aligned} \begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{bmatrix} =& \begin{bmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v & -\frac{1}{2}G_u \\ -\frac{1}{2}E_v & \frac{1}{2}G_u & \frac{1}{2}G_v \end{bmatrix} \\ =& \frac{1}{2}\begin{bmatrix} \frac{E_u}{E} & \frac{E_v}{E} & -\frac{G_u}{E} \\ -\frac{E_v}{G} & \frac{G_u}{G} & \frac{G_v}{G} \end{bmatrix} \end{aligned}$$

Again, apply the Prop 2 in lecture,

$$
\begin{aligned}
K =& \frac{1}{E}\left((\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uv}\Gamma^v_{uu} - \Gamma^v_{uv}\Gamma^v_{vu}\right)\\
=& \frac{1}{E}\left(\left(-\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u + \left(\frac{E_u}{2E}\right)\left(\frac{G_u}{2G}\right) + \left(-\frac{E_v}{2G}\right)\left(\frac{G_v}{2G}\right) - \left(\frac{E_v}{2E}\right)\left(-\frac{E_v}{2G}\right) - \left(\frac{G_u}{2G}\right)\left(\frac{G_u}{2G}\right)\right)\\
=& \frac{1}{E}\left(\left(-\frac{E_v}{2G}\right)_v + \left(-\frac{G_u}{2G}\right)_u + \frac{E_uG_u}{4EG} - \frac{E_vG_v}{4G^2} + \frac{E_vE_v}{4EG} - \frac{G_uG_u}{4G^2}\right)\\
=& \text{RHS}.
\end{aligned}
$$

Hence

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right).$$

□