PROBLEM SET 10 – §3.2

Exercise 3.2.5: Let $\phi(u, v)$ parameterize a surface in $\mathbb{R}^3$, and define a new surface by $\psi(u, v) = c \cdot \phi(u, v)$ for some fixed $c > 0$. Then:

$$\begin{aligned}
\psi_u &= c \cdot \phi_u \\
\psi_v &= c \cdot \phi_v \\
\psi_{uu} &= c \cdot \phi_{uu} \\
\psi_{uv} &= c \cdot \phi_{uv} \\
\psi_{vv} &= c \cdot \phi_{vv} \\
U_\psi &= \frac{\psi_u \times \psi_v}{|\psi_u \times \psi_v|} = \frac{(c\phi_u) \times (c\phi_v)}{|(c\phi_u) \times (c\phi_v)|} = \frac{c^2 \left(\phi_u \times \phi_v\right)}{c^2 \left|\phi_u \times \phi_v\right|} = U_\phi \\
E_\psi &= \psi_u \bullet \psi_u = c^2 \phi_u \bullet \phi_u = c^2 \cdot E_\phi \\
F_\psi &= \psi_u \bullet \psi_v = c^2 \phi_u \bullet \phi_v = c^2 \cdot G_\phi \\
G_\psi &= \psi_v \bullet \psi_v = c^2 \phi_v \bullet \phi_v = c^2 \cdot G_\phi \\
\ell_\psi &= \psi_{uu} \bullet U_\psi = c\dot{\phi}_{uu} \bullet U_\phi = c \cdot \ell_\phi \\
m_\psi &= \psi_{uv} \bullet U_\psi = c\dot{\phi}_{uv} \bullet U_\phi = c \cdot m_\phi \\
n_\psi &= \psi_{vv} \bullet U_\psi = c\dot{\phi}_{vv} \bullet U_\phi = c \cdot n_\phi \\
K_\psi &= \frac{\ell_\psi \cdot n_\psi - (m_\psi)^2}{E_\psi \cdot G_\psi - (F_\psi)^2} = \frac{c\ell_\phi \cdot cn_\phi - (cm_\phi)^2}{c^2 E_\phi \cdot c^2 G_\phi - (c^2 F_\psi)^2} = \frac{c^2 \left(\ell_\phi \cdot n_\phi - (m_\phi)^2\right)}{c^4 \left(E_\phi \cdot G_\phi - (F_\phi)^2\right)} = \frac{1}{c^2} \cdot K_\phi
\end{aligned}$$

Exercise 3.2.7: Let $M = \phi(u, v)$ be a surface, which is locally non-umbilic. We claim that all u- and v-parameter curves are lines of curvature if and only if F and m are identically zero. That is, we claim that each point we have $F = 0 = m$ if and only if ϕ_u and ϕ_v are the principle directions (and are therefore eigenvectors for the shape operator).

Assume first that ϕ_u and ϕ_v are principle directions in our non-umbilic region, so that $S(\phi_u) = k_1 \cdot \phi_u$ and $S(\phi_v) = k_2 \cdot \phi_v$ for some $k_1 \neq k_2$. Since $S(\phi_u) \bullet \phi_v = m = S(\phi_v) \bullet \phi_u$, we have:

$$m = S(\phi_u) \bullet \phi_v = k_1 (\phi_u \bullet \phi_v) \qquad \text{and}$$
$$m = S(\phi_v) \bullet \phi_u = k_2 (\phi_v \bullet \phi_u).$$

Hence:

$$0 = m - m = (k_1 - k_2)\, \phi_u \bullet \phi_v$$

Since our assumption that the surface is non-umbilic means that $k_1 \neq k_2$, we deduce that $F = \phi_u \bullet \phi_v = 0$, and hence $m = 0$ as well.

Conversely, assume now that $m = F = 0$. Writing the shape operator as $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have:

$$0 = m = S(\phi_u) \bullet \phi_v = (a\phi_u + b\phi_v) \bullet \phi_v = a \underbrace{\phi_u \bullet \phi_v}_{=F=0} + b\phi_v \bullet \phi = b\phi_v \bullet \phi_v$$
$$0 = m = S(\phi_v) \bullet \phi_u = (c\phi_u + d\phi_v) \bullet \phi_u = c\phi_u \bullet \phi_u + d \underbrace{\phi_v \bullet \phi_u}_{=F=0} = c\phi_u \bullet \phi_u$$

Since neither ϕ_u nor ϕ_v is zero, the first equation implies that $b = 0$, and the second implies that $c = 0$. Hence we have as claimed:

$$S(\phi_u) = a\phi_u \qquad \text{and} \qquad S(\phi_v) = d\phi_v$$

Exercise 3.2.13: Consider the elliptic paraboloid M which is graph in $\mathbb{R}^3$ of $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Parameterize M by:

$$\phi(u,v) = \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2}\right)$$

We compute the Gaussian curvature of M as follows:

$$\phi_u = \left(1, 0, \frac{2u}{a^2}\right) \qquad \phi_v = \left(0, 1, \frac{2v}{b^2}\right)$$

$$\phi_{uu} = \left(0, 0, \frac{2}{a^2}\right) \qquad \phi_{uv} = (0,0,0) \qquad \phi_{vv} = \left(0, 0, \frac{2}{b^2}\right)$$

$$\phi_u \times \phi_v = \left(-\frac{2u}{a^2}, -\frac{2v}{b^2}, 1\right) \qquad |\phi_u \times \phi_v| = \left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{1/2}$$

$$\begin{aligned}
U &= \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{1/2}} \cdot \left(-\frac{2u}{a^2}, -\frac{2v}{b^2}, 1\right) \\
E &= \phi_u \bullet \phi_u = 1 + \frac{4u^2}{a^4} = \frac{a^4 + 4u^2}{a^4} \\
F &= \phi_u \bullet \phi_v = \frac{4uv}{a^2 b^2} \\
G &= \phi_v \bullet \phi_v = 1 + \frac{4v^2}{b^4} = \frac{b^4 + 4v^2}{b^4} \\
\ell &= \phi_{uu} \bullet U = \frac{2}{a^2} \cdot \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{1/2}} \\
m &= \phi_{uv} \bullet U = 0 \\
n &= \phi_{vv} \bullet U = \frac{2}{b^2} \cdot \frac{1}{\left(\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1\right)^{1/2}} \\
K &= \left(\ell n - m^2\right) \cdot \left(EG - F^2\right)^{-1} \\
&= \left(\frac{4}{a^2 b^2} \cdot \frac{1}{\frac{4u^2}{a^4} + \frac{4v^2}{b^4} + 1} - 0\right) \cdot \left(\frac{a^4 + 4u^2}{a^4} \cdot \frac{b^4 + 4v^2}{b^4} - \frac{16u^2v^2}{a^4 b^4}\right)^{-1} \\
&= \frac{1}{a^2 b^2} \cdot \frac{1}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{4}} \cdot \frac{a^4 b^4}{(a^4 + 4u^2)(b^4 + 4v^2) - 16u^2v^2} \\
&= \frac{1}{a^2 b^2} \cdot \frac{1}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{4}} \cdot \frac{a^4 b^4}{a^4 b^4 + 16u^2v^2 + 4a^4v^2 + 4b^4v^2 - 16u^2v^2} \\
&= \frac{1}{a^2 b^2} \cdot \frac{1}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{4}} \cdot \frac{a^4 b^4}{a^4 b^4 + 4a^4v^2 + 4b^4v^2} \\
&= \frac{1}{a^2 b^2} \cdot \frac{1}{\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{4}} \cdot \frac{1}{4\left(\frac{1}{4} + \frac{v^2}{b^4} + \frac{u^2}{a^4}\right)} \\
&= \frac{1}{4a^2 b^2 \left(\frac{u^2}{a^4} + \frac{v^2}{b^4} + \frac{1}{4}\right)^2}
\end{aligned}$$

Exercise 3.2.18: We assume the result of part **(a)**, which says that a ruled surface $\phi(u,v) = \beta(u) + v\delta(u)$ has Gauss curvature:

$$K = \frac{-\left(\beta' \bullet \delta \times \delta'\right)^2}{W^4}$$

Here, we have $W := |\beta' \times \delta + v\delta' \times \delta|$.

(b): Let M be the saddle surface given by $z = xy$. Ruling M by $\phi(u,v) = \underbrace{(u,0,0)}_{=\beta} + v\underbrace{(0,1,u)}_{=\delta}$, we have:

$$\begin{aligned}
\beta' &= (1,0,0)\\
\delta' &= (0,0,1)\\
\beta' \times \delta &= (0,-u,1)\\
\delta' \times \delta &= (1,0,0)\\
W^4 &= |(0,-u,1) + v(1,0,0)|^4 = \left(u^2 + v^2 + 1\right)^2\\
K &= \frac{-\left((1,0,0) \bullet (-1,0,0)\right)^2}{W^4} = \frac{-1}{\left(u^2+v^2+1\right)^2}
\end{aligned}$$

(c): Now consider an arbitrary cone ruled by $\phi(u,v) = \mathbf{p} + v\delta(u)$, where $\beta = \mathbf{p}$, a constant. Since $\beta' \equiv 0$, $\boxed{\text{the curvature } K \text{ must equal zero}}$ everywhere it is defined. Observing that $W = |\beta' \times \delta + v\delta' \times \delta| = |v\delta' \times \delta|$, we see that K will be defined (and zero) everywhere except where $v = 0$ (at $\mathbf{p}$) and where $\delta' \times \delta = \mathbf{0}$. The fact that we have a regular surface implies that the latter is impossible, and hence that everywhere other than at $\mathbf{p}$ our surface has curvature $K \equiv 0$.

(d): Consider an arbitrary cylinder ruled by $\phi(u,v) = \beta(u) + v\mathbf{q}$, where $\delta = \mathbf{q}$, a constant. This gives $\delta' \equiv 0$, and hence $K = \frac{0}{W^4}$. Since we further have here that $W = |\beta' \times \delta| \neq 0$, due to the regularity of ϕ, we conclude that $\boxed{K \equiv 0}$.

(e): Consider the helicoid ruled by $\phi(u,v) = \underbrace{(0,0,cu)}_{=\beta} + v\underbrace{(R\cos u, R\sin u, 0)}_{=\delta}$. We have:

$$\begin{aligned}
\beta' &= (0,0,c)\\
\delta' &= (-R\sin u, R\cos u, 0)\\
\delta' \times \delta &= (0,0,-R^2)\\
\beta' \times \delta &= (-Rc\sin u, -Rc\cos u, 0)\\
W^4 &= \left|(-Rc\sin u, -Rc\cos u, 0) + v(0,0,-R^2)\right|^4 = \left|(-Rc\sin u, -Rc\cos u, -vR^2)\right|^4 = \left(R^2c^2 + v^2R^4\right)^2\\
K &= \frac{-\left(cR^2\right)^2}{\left(R^2c^2 + v^2R^4\right)^2} = \frac{-c^2}{\left(c^2 + v^2R^2\right)^2}
\end{aligned}$$

Note that with $R = 1$, this simplifies to:

$$K = \frac{-c^2}{\left(c^2 + v^2\right)^2}$$