Introduction to Differential Geometry I
Homework 1

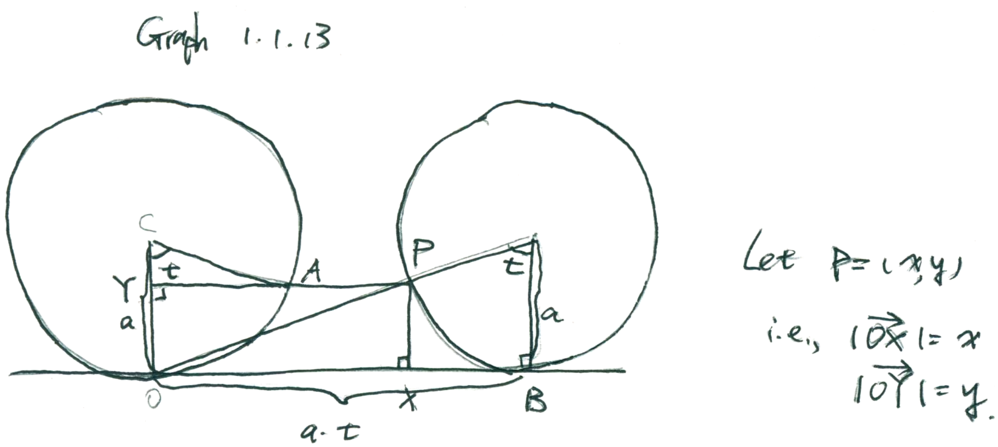
MATH:4500

September 6, 2017

Problem 1. 1.1.13

Solution:

From the hint of the graph,



The coordinate of $P = (x, y)$ is given by:

$$x = |\overrightarrow{OX}| = |\overrightarrow{OB}| - |\overrightarrow{XB}| = a \cdot t - a \sin t$$

$$y = |\overrightarrow{OY}| = |\overrightarrow{OC}| - |\overrightarrow{YC}| = a - a \cos t$$

Therefore, $(x, y) = (at - a \sin t, a - a \cos t)$. i.e.,

$$\alpha(t) = (a\,(t - \sin(t), a\,(1 - \cos(t))\,.$$

□

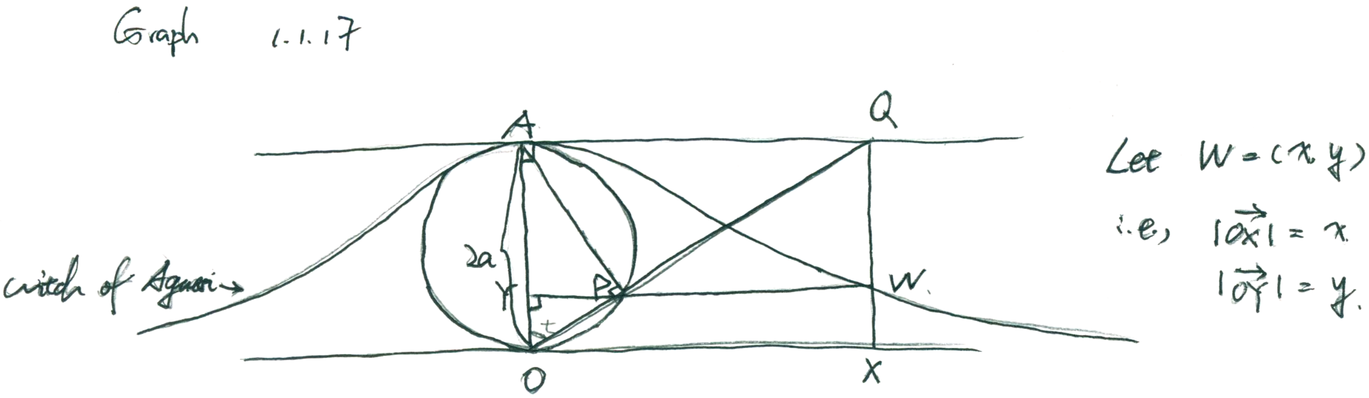
Problem 2. 1.1.17

Solution:

See the graph, the point on the *witch of Agnesi* is denoted as W in the graph.

Suppose its coordinate is $W = (x, y)$, i.e., $|\overrightarrow{OX}| = x$, $|\overrightarrow{OY}| = y$, so we need to computer the length of these vectors to have the value of x and y.

Connect the point A, Y with P.



Note that the points Y, P, W is on the same line $\overrightarrow{PW}$ is a horizontal line as we defined, same reasoning for the points Q, W, X are on the same vertical line.

Since $\overrightarrow{OA}$ is the diagram of the circle of radius a centered at $(0, a)$, then $\triangle APO$ is a right triangle, which with $\angle APO = \frac{\pi}{2}, \angle AOP = t$.

In addition, $\triangle APY$ is also a right triangle, which with $\angle PYO = \frac{\pi}{2}, \angle YOP = t$.

Therefore,

$$\begin{aligned} y = |\overrightarrow{OY}| &= |\overrightarrow{OP}| \cdot \cos(t) && \text{(in } \triangle \text{ PYO)} \\ &= |\overrightarrow{OA}| \cdot \cos(t) \cdot \cos(t) && \text{(in } \triangle \text{ APO)} \\ &= 2a \cos^2(t). \end{aligned}$$

And if we consider the right triangle $\triangle OAQ$ which with $\angle OAQ = \frac{\pi}{2}, \angle AOQ = t$, we can derive:

$$\begin{aligned} x = |\overrightarrow{OX}| &= |\overrightarrow{AQ}| \\ &= |\overrightarrow{OA}| \cdot \tan(t) \\ &= 2a \tan(t). \end{aligned}$$

As a result, we have

$$W(t) = (x, y) = (2a\ \tan(t),\ 2a\ \cos^2(t))$$

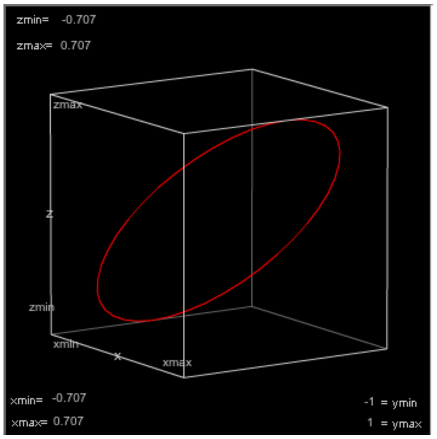
□

Problem 3. 1.1.25

Solution: Since $|\alpha(t)| = \sqrt{\left[\frac{1}{\sqrt{2}}\cos(t)\right]^2 + \sin^2(t) + \left[\frac{1}{\sqrt{2}}\cos(t)\right]^2} = 1,$

In fact, since $\alpha(t) = v\cos t + w\sin t$ where $v = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, and $w = (0, 1, 0)$.

thus this curve $\alpha(t)$ is a circle lies on the plane $x = z$. Also shown by the graph below:



And we have,

$$\alpha'(t) = \left(-\frac{1}{\sqrt{2}}\sin(t), \cos(t), -\frac{1}{\sqrt{2}}\sin(t)\right), \alpha''(t) = \left(\frac{1}{\sqrt{2}}\cos(t)\right) + \sin(t) + \left(\frac{1}{\sqrt{2}}\cos(t)\right)$$

In addition,

$$\begin{aligned} L(\alpha) &= \int_0^{2\pi} |\alpha'(t)| dt \\ &= \int_0^{2\pi} \sqrt{\left[-\frac{1}{\sqrt{2}}\sin(t)\right]^2 + \cos^2(t) + \left[-\frac{1}{\sqrt{2}}\sin(t)\right]^2}\, dt \\ &= \int_0^{2\pi} dt \\ &= 2\pi \end{aligned}$$

□

Problem 4. 1.1.26

Solution:

If we move the center point of the our circle (a, b) to the origin $(0, 0)$, then we will derive the unit circle. And every point (x, y) on the circle is move to the point $((x - a), (y - b))$.

Let $\hat{x} = x - a$, $\hat{y} = y - b$, then the points $(\hat{x}, \hat{y})$ are on the unit circle.

For which has a parameterized equation

$$\hat{x} = \cos(t)\,, \quad \hat{y} = \sin(t)\,.$$

Therefore, we have the parameterization for the circle center at (a, b) as:

$$x = a + \cos(t)\,, \quad y = b + \sin(t)\,.$$

i.e.,

$$\alpha(t) = (a + \cos(t)\,, b + \sin(t)\,).$$

□

Problem 5. 1.1.27

Solution:

Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, i.e., we have $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

Denote $\hat{x} = \frac{x}{a}$, $\hat{y} = \frac{y}{b}$, therefore,

$$\hat{x}^2 + \hat{y}^2 = 1.$$

And we know the parameterization of a circle as

$$\hat{x} = \cos(t)\,, \quad \hat{y} = \sin(t)\,.$$

Thus, we derive the parameterization of a ellipse as:

$$x = a\cos(t)\,, \quad y = b\sin(t)\,.$$

i.e.,

$$\alpha(t) = (a\cos(t)\,, b\sin(t)\,).$$

□