MATH 2850
MIDTERM 2
November 11, 2016

NAME. SOLUTION

SIGNATURE.________________________

Do all **5** problems, **20 points each.**

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. Show all of your work or indicate its location in the space provided after each problem. If the question is asking to provide an exact answer, (for example: $\sqrt{2}$, $\ln 2$, e^3 or $\sin\frac{\pi}{8}$), then providing a decimal answer obtained from a calculator will not receive full credit. Only writing a final answer of a question may not receive full credit, unless it is indicated otherwise. You need to indicate the steps of procedures and show the details of your work to receive full credit.

If you have any questions, please ask your proctor, do not guess. Please put away your cell phones (turn them off), laptops, textbooks and notes.

DO NOT WRITE BELOW:

1.______

2.______

3.______

4.______

5.______

Total.______

Problem 1.
Let $F(x,y,z) = x^3 - y^3 + z^3 + xyz$
a. Calculate ∇F.

$$\nabla F = (3x^2 + yz,\ -3y^2 + xz,\ 3z^2 + xy)$$

b. Calculate the directional derivative of F at $(x,y,z) = (2,1,0)$ in the direction of the vector $3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$.

$$\nabla F(2,1,0) = (12, -3, 2)$$

$$u = \frac{3i + 2j + 6k}{\sqrt{9+4+36}} = \frac{1}{7}(3i + 2j + 6k)$$

$$(D_u F)(2,1,0) = u \cdot \nabla F(2,1,0)$$

$$= \frac{1}{7}(3,2,6) \cdot (12,-3,2) = \frac{1}{7}(36 - 6 + 12) = 6$$

c. In which direction does F increase fastest at $(2,1,0)$?

$$\frac{\nabla F(2,1,0)}{\|\nabla F(2,1,0)\|} = \frac{(12,-3,2)}{\sqrt{144+9+4}} = \frac{1}{\sqrt{157}}(12,-3,2)$$

d. Find an equation for tangent plane to the level set $x^3 - y^3 + z^3 + xyz = 7$ at $(2,1,0)$.

$$(12,-3,2) \cdot (x-2,\ y-1,\ z-0) = 0$$

$$12x - 3y + 2z = 24 - 3 + 0 = 21$$

Problem 2. Let $\mathbf{x}(t) = (t, t^2, \frac{2}{3}t^3)$ for $1 \le t \le 3$.

a. For the path defined by $\mathbf{x}(t)$, find its velocity, acceleration and speed. Indicate which is which.

velocity $x' = (1, 2t, 2t^2)$

speed $|x'| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$

acceleration $x'' = (0, 2, 4t)$

b. Find a parametric equation for the tangent line to $\mathbf{x}(t)$ when $t = 2$.

$x(2) = (2, 4, \frac{16}{3})$

$x'(2) = (1, 4, 8)$

$\ell(s) = (2, 4, \frac{16}{3}) + s(1, 4, 8)$

OR

$= (2, 4, \frac{16}{3}) + (s-2)(1, 4, 8)$

c. Find the length of the path given by $\mathbf{x}(t)$.

$$\int_1^3 |x'(t)|\,dt = \int_1^3 (1 + 2t^2)\,dt = t + \frac{2}{3}t^3 \Big|_1^3$$

$$= (3 + 18) - (1 + \frac{2}{3}) = \frac{58}{3}$$

Problem 3.

a. Evaluate $\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} 6y\, dz\, dy\, dx.$

$$= \int_{-1}^{1}\int_{x^2}^{1} 6yz \Big|_{z=0}^{z=1-y} dy\,dx$$

$$= \int_{-1}^{1}\int_{x^2}^{1} 6y(1-y)\, dy\, dx = \int_{-1}^{1}\int_{x^2}^{1} (6y - 6y^2)\, dy\, dx$$

$$= \int_{-1}^{1} 3y^2 - 2y^3 \Big|_{y=x^2}^{y=1} dx = \int_{-1}^{1} \left((3-2) - (3x^4 - 2x^6)\right) dx$$

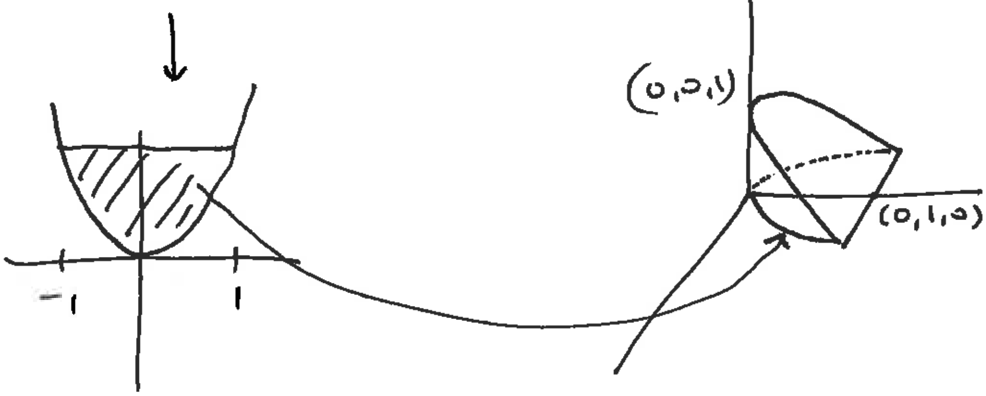
$$= \int_{-1}^{1} (1 - 3x^4 + 2x^6)\, dx = x - \frac{3}{5}x^5 + \frac{2}{7}x^7 \Big|_{-1}^{1}$$

$$= \left(1 - \frac{3}{5} + \frac{2}{7}\right) - \left(-1 + \frac{3}{5} - \frac{2}{7}\right) = 2 \cdot \frac{35 - 21 + 10}{35} = \frac{48}{35}$$

b. Sketch the domain of integration of the integral above.

$-1 \le x \le 1$
$x^2 \le y \le 1$

$-1 \le x \le 1$
$x^2 \le y \le 1$
$0 \le z \le 1-y$



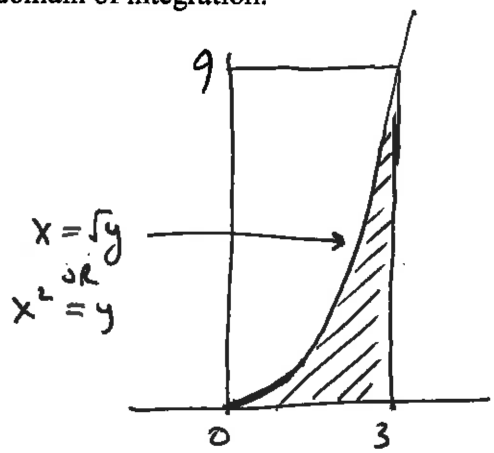
Problem 4.

Consider the integral $\int_0^9 \int_{\sqrt{y}}^3 3x^3 \cos(xy)\, dx\, dy$.

a. Sketch the domain of integration.

$0 \le y \le 9$

$\sqrt{y} \le x \le 3$



b. Reverse the order of integration of the integral above.

$$\int_0^3 \int_0^{x^2} 3x^3 \cos(xy)\, dy\, dx$$

c. Find the exact (numerical) value of the integral above.

Since $\frac{\partial}{\partial y} \sin xy = x \cos xy$

$$I = \int_0^3 \int_0^{x^2} \underbrace{3x^2 \cdot x}_{3x^3} \cos(xy)\, dy\, dx$$

$$= \int_0^3 3x^2 \sin xy \Big|_{y=0}^{y=x^2} = \int_0^3 \Big(3x^2 \sin x^3 - \underbrace{3x^2 \sin 0}_{0}\Big)\, dx$$

$u = x^3$

$du = 3x^2\, dx$

$$I = \int_0^{27} \sin u\, du = -\cos u \Big|_0^{27} = -\cos 27 + \underbrace{\cos 0}_{1} = 1 - \cos 27$$

Problem 5. Let $f(x,y) = -x^2 + y^2 - 4y : \mathbf{R}^2 \to \mathbf{R}$.

a. Find all critical points of f. Identify the nature of each of these critical points (local maxima, local minima, saddle points).

$\nabla f = (-2x, 2y-4)$

$\nabla f = 0 \iff \begin{cases} -2x = 0 \\ 2y-4=0 \end{cases}$

$\iff x = 0$ & $y = 2$

$(0,2)$ c.p.

$H_f = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

$d_1 = -2$

$d_2 = -4 < 0 \implies (0,2)$ is a saddle pt.

b. Find the absolute maximum and minimum values of $f(x,y)$ on the closed disk $\{(x,y) \mid x^2 + y^2 \le 9\}$.

We first take care of the boundary $x^2+y^2=9$.

Let $g = x^2+y^2$

$\nabla g = (2x, 2y)$ $\quad \nabla f = (-2x, 2y-4)$

$\nabla f = \lambda \nabla g \implies$
$$-2x = 2\lambda x \quad (1)$$
$$2y-4 = 2\lambda y \quad (2)$$
$$x^2+y^2 = 9 \quad (3)$$

(1) $2x + 2\lambda x = 0$

$2x(1+\lambda) = 0$

$x = 0$ OR $\lambda = -1$

(3) $y^2 = 9$

$y = \pm 3$

c.p.

$(0,3)$

$(0,-3)$

(2) $2y - 4 = -2y$

$4y = 4$

$y = 1$

(3) $x^2 + 1 = 9$

$x^2 = 8$

$x = \pm\sqrt{8}$

c.p

$(\sqrt{8}, 1)$

$(-\sqrt{8}, 1)$

		$-x^2+y^2-4y$
interior c.p.	$(0,2)$	$4-8=-4$
boundary c.p.	$(0,3)$	$9-12=-3$
	$(0,-3)$	$9+12=21$
	$(\sqrt{8},1)$	$-8+1-4=-11$
	$(-\sqrt{8},1)$	$-8+1-4=-11$

max is 21 at $(0,-3)$

min is -11 at $(\pm\sqrt{8}, 1)$.