

**MATH 2850
MIDTERM 2
November 11, 2016**

NAME. _____

SOLUTION

SIGNATURE. _____

Do all 5 problems, 20 points each.

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. Show all of your work or indicate its location in the space provided after each problem. If the question is asking to provide an exact answer, (for example: $\sqrt{2}$, $\ln 2$, e^3 or $\sin \frac{\pi}{8}$), then providing a decimal answer obtained from a calculator will not receive full credit. Only writing a final answer of a question may not receive full credit, unless it is indicated otherwise. You need to indicate the steps of procedures and show the details of your work to receive full credit.

If you have any questions, please ask your proctor, do not guess. Please put away your cell phones (turn them off), laptops, textbooks and notes.

DO NOT WRITE BELOW:

1. _____

2. _____

3. _____

4. _____

5. _____

TOTAL. _____

Problem 1.

Let $F(x,y,z) = x^3 - y^3 + z^3 + xyz$

a. Calculate ∇F .

$$\nabla F = (3x^2 + yz, -3y^2 + xz, 3z^2 + xy)$$

b. Calculate the directional derivative of F at $(x,y,z) = (1,0,2)$ in the direction of the vector $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

$$\nabla F(1,0,2) = (3, 2, 12)$$

$$\mathbf{u} = \frac{3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}}{\sqrt{9 + 4 + 36}} = \frac{1}{7} (3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k})$$

$$\begin{aligned} (D_{\mathbf{u}} F)(1,0,2) &= \mathbf{u} \cdot \nabla F(1,0,2) = \frac{1}{7} (3, -2, 6) \cdot (3, 2, 12) \\ &= \frac{1}{7} (9 - 4 + 72) = 11 \end{aligned}$$

c. In which direction does F increase fastest at $(1,0,2)$?

$$\text{in the } \frac{\nabla F(1,0,2)}{|\nabla F(1,0,2)|} \text{ direction } \quad \frac{(3, 2, 12)}{\sqrt{9 + 4 + 144}} = \frac{1}{\sqrt{157}} (3, 2, 12)$$

d. Find an equation for tangent plane to the level set $x^3 - y^3 + z^3 + xyz = 9$ at $(1,0,2)$.

$$(3, 2, 12) \cdot (x-1, y-0, z-2) = 0$$

$$3x + 2y + 12z = 3 + 0 + 24 = 27.$$

Problem 2. Let $\mathbf{x}(t) = (2t, t^2, \frac{1}{3}t^3)$ for $1 \leq t \leq 3$.

a. For the path defined by $\mathbf{x}(t)$, find its velocity, acceleration and speed. Indicate which is which.

$$\text{velocity } \mathbf{x}' = (2, 2t, t^2)$$

$$\text{speed } |\mathbf{x}'| = \sqrt{4 + 4t^2 + t^4} = 2 + t^2$$

$$\text{acceleration } \mathbf{x}'' = (0, 2, 2t)$$

b. Find a parametric equation for the tangent line to $\mathbf{x}(t)$ when $t = 2$.

$$\mathbf{x}(2) = (4, 4, \frac{8}{3})$$

$$\mathbf{x}'(2) = (2, 4, 4)$$

$$\ell(s) = (4, 4, \frac{8}{3}) + s(2, 4, 4)$$

OR

$$(4, 4, \frac{8}{3}) + (s-2)(2, 4, 4)$$

c. Find the length of the path given by $\mathbf{x}(t)$.

$$\int_1^3 |\mathbf{x}'(t)| dt = \int_1^3 (2 + t^2) dt = 2t + \frac{t^3}{3} \Big|_1^3$$

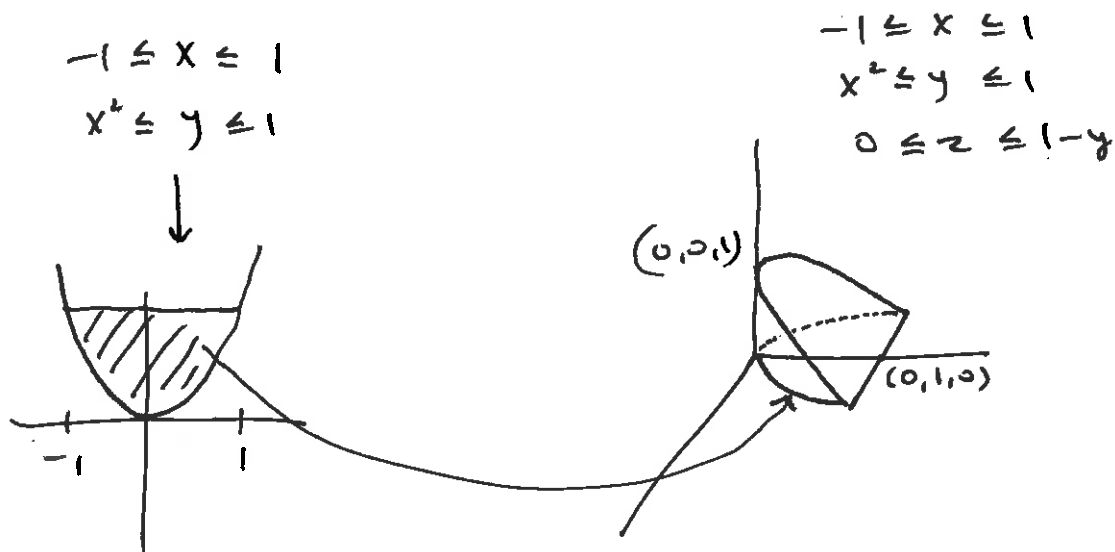
$$= (6 + 9) - (2 + \frac{1}{3}) = \frac{38}{3}$$

Problem 3.

a. Evaluate $\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} 6y \, dz \, dy \, dx$.

$$\begin{aligned}
 &= \int_{-1}^1 \int_{x^2}^1 6yz \Big|_{z=0}^{z=1-y} dy \, dx \\
 &= \int_{-1}^1 \int_{x^2}^1 6y(1-y) dy \, dx = \int_{-1}^1 \int_{x^2}^1 (6y - 6y^2) dy \, dx \\
 &= \int_{-1}^1 3y^2 - 2y^3 \Big|_{y=x^2}^{y=1} dx = \int_{-1}^1 ((3-2) - (3x^4 - 2x^6)) dx \\
 &= \int_{-1}^1 (1 - 3x^4 + 2x^6) dx = x - \frac{3}{5}x^5 + \frac{2}{7}x^7 \Big|_{-1}^1 \\
 &= \left(1 - \frac{3}{5} + \frac{2}{7}\right) - \left(-1 + \frac{3}{5} - \frac{2}{7}\right) = 2 \cdot \frac{35 - 21 + 10}{35} = \frac{48}{35}
 \end{aligned}$$

b. Sketch the domain of integration of the integral above.



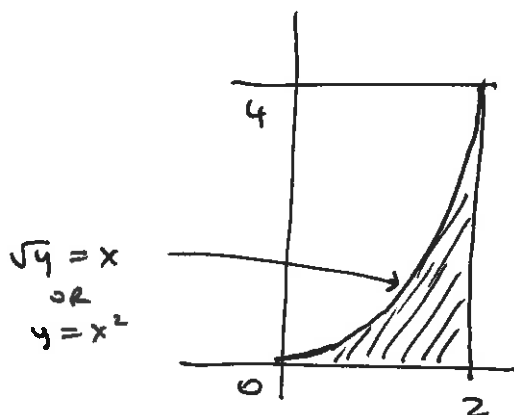
Problem 4.

Consider the integral $\int_0^4 \int_{\sqrt{y}}^2 3x^3 \cos(xy) dx dy$.

a. Sketch the domain of integration.

$$0 \leq y \leq 4$$

$$\sqrt{y} \leq x \leq 2$$



b. Reverse the order of integration of the integral above.

$$\int_0^2 \int_0^{x^2} 3x^3 \cos xy dy dx.$$

c. Find the exact (numerical) value of the integral above.

since $\frac{d}{dy} \sin xy = x \cos xy$

$$\int_0^2 \int_0^{x^2} \underbrace{3x^2 \cdot x}_{3x^3} \cos xy dy dx = \int_0^2 3x^2 \sin xy \Big|_{y=0}^{y=x^2} dx$$

$$= \int_0^2 \left(3x^2 \sin x^3 - \underbrace{3x^2 \sin 0}_0 \right) dx = \int_0^8 \sin u du$$

$$u = x^3$$

$$du = 3x^2 dx$$

$$= -\cos u \Big|_0^8 = -\cos 8 + \underbrace{\cos 0}_1 = 1 - \cos 8$$

Problem 5. Let $f(x,y) = 4x - x^2 + y^2 : \mathbb{R}^2 \rightarrow \mathbb{R}$.

a. Find all critical points of f . Identify the nature of each of these critical points (local maxima, local minima, saddle points).

$$\nabla f = (4 - 2x, 2y) \quad \nabla f = 0 \iff \begin{cases} 4 - 2x = 0 \\ 2y = 0 \end{cases} \iff \begin{cases} x = 2 \\ y = 0 \end{cases}$$

$$H_f = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \quad n=2 \quad d_1 = -2 \quad d_2 = -4 < 0 \implies (2,0) \text{ is a saddle point.}$$

b. Find the absolute maximum and minimum values of $f(x,y)$ on the closed disk

$\{(x,y) \mid x^2 + y^2 \leq 9\}$. We first take care of the boundary:

Let $g(x,y) = x^2 + y^2$

$$x^2 + y^2 = 9$$

$$\nabla g = (2x, 2y) \quad , \quad \nabla f = (4 - 2x, 2y)$$

$$\nabla f = \lambda \nabla g \implies \begin{cases} 4 - 2x = 2\lambda x & \textcircled{1} \\ 2y = 2\lambda y & \textcircled{2} \\ x^2 + y^2 = 9 & \textcircled{3} \end{cases}$$

$$\textcircled{2} \quad 2y - 2\lambda y = 0$$

$$2y(1 - \lambda) = 0$$

$$y = 0 \quad \text{or}$$

$$\lambda = 1$$

$$\textcircled{3} \quad x^2 = 9$$

$$x = \pm 3$$

c.p.:

$$\begin{pmatrix} 3, 0 \\ -3, 0 \end{pmatrix}$$

$$\textcircled{1} \quad 4 - 2x = 2x$$

$$4 = 4x$$

$$x = 1$$

$$\textcircled{3} \quad y^2 + 1 = 9$$

$$y^2 = 8$$

$$y = \pm \sqrt{8}$$

c.p.

$$\begin{pmatrix} 1, \sqrt{8} \\ 1, -\sqrt{8} \end{pmatrix}$$

boundary c.p.

	$4x - x^2 + y^2$
$(2,0)$	$8 - 4 = 4$
$(3,0)$	$12 - 9 = 3$
$(-3,0)$	$-12 - 9 = -21$
$(1, \sqrt{8})$	$4 - 1 + 8 = 11$
$(1, -\sqrt{8})$	$4 - 1 + 8 = 11$

Max 11 at $(1, \pm\sqrt{8})$

min -21 at $(-3,0)$

interior c.p.