MATH:2850
MIDTERM 1
September 30, 2016

NAME. SOLUTION

SIGNATURE.________________________

Do all **5** problems, **20 points each.**

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. Show all of your work or indicate its location in the space provided after each problem. If the question is asking to provide an exact answer, (for example: $\sqrt{2}$, $\ln 2$, e^3 or $\sin\frac{\pi}{8}$), then providing a decimal answer obtained from a calculator will not receive full credit. Only writing a final answer of a question may not receive full credit, unless it is indicated otherwise. You need to indicate the steps of procedures and show the details of your work to receive full credit.

If you have any questions, please ask your proctor, do not guess. Please put away your cell phones (turn them off), laptops, textbooks and notes.

DO NOT WRITE BELOW:

1.______

2.______

3.______

4.______

5.______

TOTAL.______

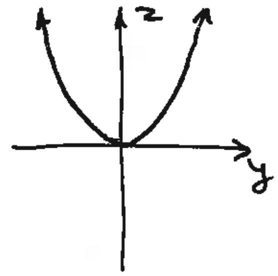
Problem 1.

Let $f(x,y) = y^2 - x^2$.

For all of the graphs below: label the axes, and label the graphs with their functions.

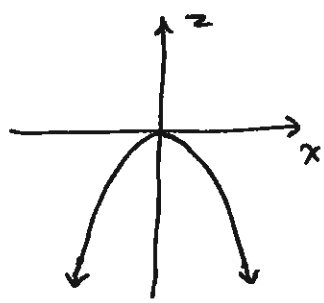
a. Sketch the section of the graph of f by the plane $x = 0$, that is $z = f(0,y)$.

$z = f(0,y) = y^2$



b. Sketch the section of the graph of f by the plane $y = 0$, that is $z = f(x,0)$.

$z = f(x,0) = -x^2$

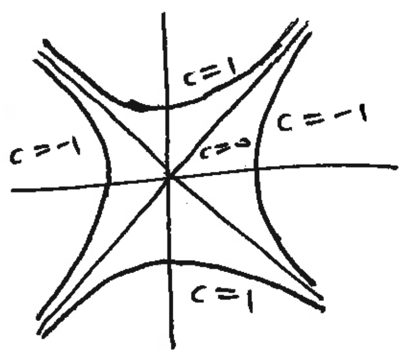


c. Sketch the level curves at height c, that is $f(x,y) = c$ for the values of $c = -1, 0$ and 1.

$y^2 - x^2 = 0 \iff y = \pm x$

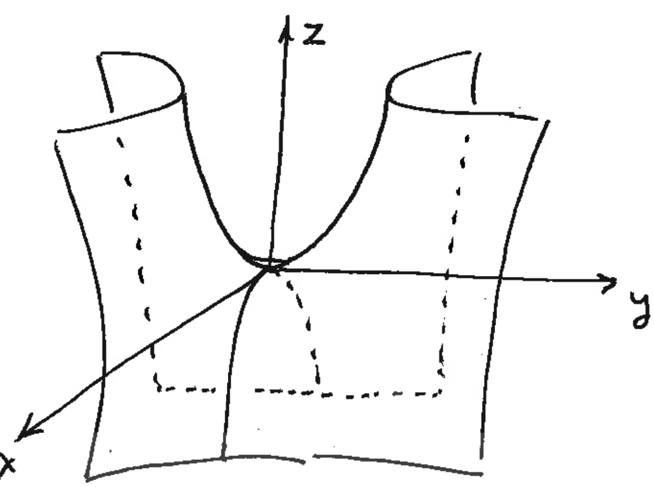
Hyperbolas $\begin{cases} y^2 - x^2 = 1 \\ y^2 - x^2 = -1 \end{cases}$

asymptotic to $y = \pm x$.



d. Sketch the explicit graph of $z = f(x,y)$. Describe it in words if you can't draw it.

$z = f(x,y) = y^2 - x^2$



It is a saddle.

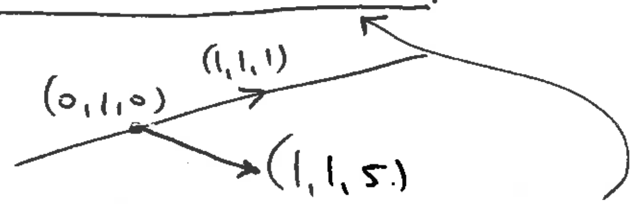
Problem 2.

Let P be the point $(1,1,5)$ and the line ℓ be given by the equations $\begin{cases} x = t \\ y = 1+t \\ z = t \end{cases}$ in $\mathbb{R}^3$.

a. Find a parametric equation for the plane that contains the point P and the line ℓ.

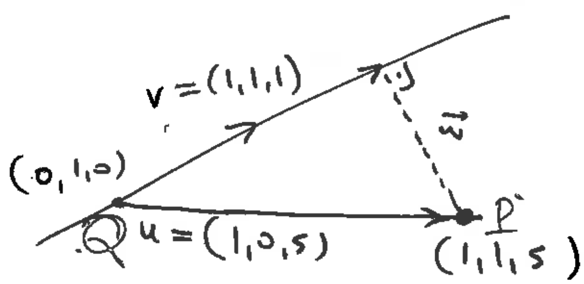
line $\ell(t) = (0,1,0) + t\,(1,1,1)$

$$\boxed{\text{plane} \quad r(s,t) = (0,1,0) + t\,(1,1,1) + s\,(1,0,5)}$$



$(1,1,5) - (0,1,0) = (1,0,5)$

b. Find the distance between P and the line ℓ.



$u = P - Q = (1,1,5) - (0,1,0)$
$= (1,0,5)$

$$\text{proj}_v u = \text{proj}_{(1,1,1)}(1,0,5) = \frac{(1,0,5)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}\,(1,1,1)$$

$$= \frac{6}{3}(1,1,1) = (2,2,2)$$

$$w = u - \text{proj}_v u = (1,0,5) - (2,2,2) = (-1,-2,3)$$

$\|w\|$ = distance between ℓ & P

$$\|w\| = \sqrt{1+4+9} = \sqrt{14}.$$

Problem 3. Let $f(x,y) = 2xye^{2y} - (\ln x) + y^3 + 3$.

a. Calculate all first and second order partial derivatives of f.

$f(1,0) = 0 - \cancel{\ln 1}^{0} + 0 + 3$

$f_x(1,0) = -1$

(for b)

$f_y(1,0) = 2$

$f_x = 2ye^{2y} - \frac{1}{x}$

$f_y = 2xe^{2y} + 2xy \cdot 2e^{2y} + 3y^2$

$f_{xx} = +\frac{1}{x^2}$

$f_{xy} = 2e^{2y} + 2y \cdot 2e^{2y} = 2e^{2y} + 4ye^{2y}$

$f_{yx} = 2e^{2y} + 4ye^{2y}$

$f_{yy} = 2x \cdot 2e^{2y} + 2xe^{2y} \cdot 2 + 2xy \cdot 4e^{2y} + 6y$

$= 8xe^{2y} + 8xye^{2y} + 6y$

b. Calculate the gradient $\nabla f(1,0)$.

$\nabla f(1,0) = (-1, 2)$

c. Find an equation describing the tangent plane to the explicit graph of $z = f(x,y)$ when $x = 1$ and $y = 0$.

$z = 3 - 1(x-1) + 2(y-0)$

Final Answer: $\boxed{z = -x + 2y + 4}$ Tangent plane equation.

$h(x,y) = -x + 2y + 4$ is the tangent plane approximation for (c)

$g(2,1) = (1,2,3)$

Problem 4.
Let $\mathbf{f}(x,y,z) = (xy - z,\ x + 2y + z^2)$ and $\mathbf{g}(u,v) = (u - v^2,\ uv,\ u + v)$.
Calculate the following: $D\mathbf{f}, D\mathbf{g}$, and $D(\mathbf{f} \circ \mathbf{g})(2,1)$.

$$D\mathbf{f} = \begin{bmatrix} y & x & -1 \\ 1 & 2 & 2z \end{bmatrix}$$

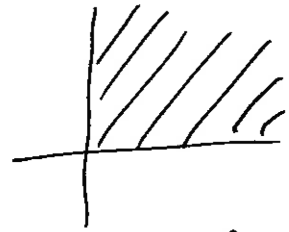
$$D\mathbf{g} = \begin{bmatrix} 1 & -2v \\ v & u \\ 1 & 1 \end{bmatrix}$$

$$D(\mathbf{f} \circ \mathbf{g})(2,1) = Df(\underbrace{g(2,1)}_{(1,2,3)}) \cdot Dg(2,1)$$

$$= Df(1,2,3) \cdot Dg(2,1)$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 9 & 8 \end{bmatrix}$$

Problem 5.



a. Let $A = \{(x,y) |\ 0 \leq x \text{ and } 0 \leq y\}$ be a subset of $\mathbb{R}^2$.
What is the boundary of A?

Boundary of $A = \{(x,y) | (0 \leq x \text{ and } y=0) \text{ OR } (0 \leq y \text{ and } x=0)\}$

Is A an open set? Circle the correct answer: YES or NO.

$(\mathrm{Bd}\, A) \cap A \neq \emptyset$, actually $(\mathrm{Bd}\, A) \subseteq A$

Is A a closed set? Circle the correct answer: YES or NO

b. Calculate the following limits. If any of them does not exist, state it so. **Justify your answers.**

i. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$ since

$$0 \leq x^2 \leq x^2+y^2$$

$$0 \leq \frac{x^2}{x^2+y^2} \leq 1$$

$$0 \leq \frac{x^2y^2}{x^2+y^2} \leq y^2$$

$y^2 \to 0$ since $(x,y) \to (0,0)$

Then use squeeze Thm.

ii. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} =$ DNE

If one approaches $(0,0)$ along x-axis : $(x,0) \to (0,0)$

$$\lim_{(x,0)\to(0,0)} \frac{x^2-0^2}{x^2+0} = 1 \quad (*)$$

If one approaches $(0,0)$ along y-axis : $(0,y) \to (0,0)$

$$\lim_{(0,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = -1 \quad (**)$$

If $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ were to exist, then $(*)$ and $(**)$ would have been equal, but they aren't. So $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE.