EULERIAN CHAINS

**Definition** A chain which goes through all edges of a graph $G$, and uses each edge of $G$ at most once is called an **Eulerian Chain**. An **Eulerian Closed Chain** is an Eulerian Chain which is closed.

**Remark** An Eulerian Closed Chain need **NOT** be a circuit, i.e. simple closed. That is, an Eulerian chain may pass through some vertices (in fact all with degree $\geq 4$) more than once. If "Eulerian Circuit" or "Eulerian Cycle" is used inadvertently in the notes, then they must be replaced by "Eulerian Closed Chain" or "Eulerian Closed Path", respectively.

**Theorem** (EULER) Let $G$ be a connected multi-graph, with no loops.

(a) There exists an Eulerian Closed Chain in $G \Leftrightarrow$ Degrees of all vertices of $G$ are even.

(b) There exists an Eulerian Chain in $G \Leftrightarrow$ The number of odd degree vertices in $G$ is either 0 or 2.

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EULERIAN CLOSED CHAIN ALGORITHM

**GIVEN:** A connected graph $G$ with all even degree vertices

**TO FIND:** An Eulerian Closed Chain, that is a closed chain which goes through all edges of the graph, and uses each edge of the graph at most once.

1. Choose any vertex, and label it $u_0$. Choose any edge starting at $u_0$, label it $E_1$ and go to the other end point $v_1$ of $E_1$.

   If there is no such edge $E_1$, then $G$ is a single point, and process is finished.

2. Assume that first $k$ edges $E_1, E_2, \ldots, E_k$ of an Eulerian chain has been constructed, which is starting at $u_0$; passing through $v_1, v_2, \ldots v_{k-1}$ and ending at $v_k$.

   Go to end point $v_k$ of $E_k$, and consider all of the remaining unlabeled edges starting at $v_k$:

   If there is an unlabeled edge, choose one, label it $E_{k+1}$ and its other end point $v_{k+1}$, and repeat step 2, until it can't be done anymore.

   If there is not any unlabeled edges left starting at $v_k$, then $v_k$ must be the initial vertex $u_0$. Label the closed chain $E_1, E_2, \ldots E_k$ with $C_1$ and go to step 3.

   The reason that this process must go on through any vertex but the initial vertex, is that all degrees are even, and any time one passes through a vertex, the number of unlabeled edges touching that vertex goes down by 2. Hence, if it is possible to reach a vertex other than the initial vertex, then there is a way out of that vertex.

3. If $C_1$ used all of the edges of $G$, then the process is finished.

   If $C_1$ did not use all of the edges of $G$, then let $G_2$ be the graph obtained from $G$ by removing all the edges (not the vertices) used by $C_1$ and then removing all isolated vertices.

   At this point, we know that $C_1 \cap G_2 \neq \emptyset$, since otherwise $C_1 \cap G_2 = \emptyset$ implies that $G$ has at least 2 components, $C_1$ and $G_2$. However, recall that $G$ is connected. Caution: $G_2$ may not be connected.

4. Choose any vertex $u_1$ of $C_1 \cap G_2$.

   On $G_2$ starting from the initial vertex $u_1$ repeat steps 1, 2 (several times) until you reach step 3.

   Label the new closed chain (from $u_1$ to itself) you obtained with $C_2$.

   Construct a closed chain $C_2$ as follows: Start at $u_0$. Follow $C_1$ to $u_1$, follow all of $C_2$ starting and ending at $u_0$, follow the rest of $C_1$ to $u_0$.

5. If $C_2$ used all of the edges of $G$, then the process is finished.

   If $C_2$ did not use all of the edges of $G$, then let $G_3$ be the graph obtained from $G$ by removing all the edges (not the vertices) used by $C_2$ and then removing all isolated vertices. Then repeat step 4, by choosing a vertex $u_2$ of $C_2 \cap G_2$ and so on. Repeat step 5, until you obtain an Eulerian chain $C_n$ which uses all edges of $G$.

   This process must finish after a finite repetition of step 5, since $0 \leq e(G_3) < e(G_2) < e(G) < \infty$, where $e(G)$ is the number of edges of $G$. 

For each of the following graphs (or digraphs): 

(a) Find an Eulerian circuit if there is one, 
(b) " " " chain if " " " or 
(c) If neither is possible, state it so & explain why.

1. 

2. 

3. 

4. 

5. 

6. 

7. 