Nonuniform Thickness and Weighted Distance

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Abstract. Nonuniform tubular neighborhoods of curves in $\mathbf{R}^n$ are studied by using weighted distance functions and generalizing the normal exponential map. Different notions of injectivity radii are introduced to investigate singular but injective exponential maps. A generalization of the thickness formula is obtained for nonuniform thickness. All singularities within almost injectivity radius are classified by the Horizontal Collapsing Property. Examples are provided to show the distinction between the different types of injectivity radii, as well as showing that the standard differentiable injectivity radius fails to be upper semicontinuous on a singular set of weight functions.

1. Introduction

The uniform thickness of a knotted curve is the radius of the largest tubular neighborhood around the curve without intersections of the normal discs. This is also known as the normal injectivity radius IR of the normal exponential map of the curve K in the Euclidean space $\mathbf{R}^n$. The ideal knots are the embeddings of S^1 into $\mathbf{R}^3$, maximizing IR in a fixed isotopy (knot) class of fixed length. As noted in [13], "...the average shape of knotted polymeric chains in thermal equilibrium is closely related to the ideal representation of the corresponding knot type". Uniform thickness has been studied extensively in several articles including [1] G. Buck and J. Simon, [2] J. Cantarella, R. B. Kusner, and J. M. Sullivan, [5] Y. Diao, [7, 8, 9] O. C. Durumeric, [10] O. Gonzales and R. de La Llave, [11] O. Gonzales and H. Maddocks, [13] V. Katrich, J. Bendar, D. Michoud, R.G. Scharein, J. Dubochet and A. Stasiak, [14] A. Litherland, J. Simon, O. Durumeric and E. Rawdon, and [16] A. Nabutovsky. The following thickness formula was obtained earlier in [14] in the smooth case, and in [2] for $C^{1,1}$ curves in $\mathbf{R}^3$.

UNIFORM THICKNESS FORMULA *[7, Theorem 1]*

For every complete smooth Riemannian manifold M^n and every compact $C^{1,1}$ submanifold K^k ($\partial K = \emptyset$) of M,

$$IR(K, M) = \min\{FocRad(K), \frac{1}{2}DCSD(K)\}.$$

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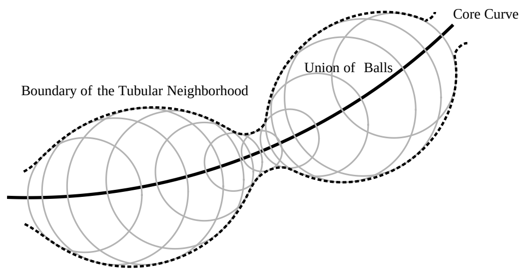

FIGURE 1. A non-uniform μr-neighborhood is shown as a union of balls of radii $r\mu(s)$ centered at $\gamma(s)$ on the core γ.

Gonzales and Maddocks [11, p. 4771] obtained that a smooth ideal knot can be partitioned into arcs of constant (maximal) global curvature and line segments. This result was later generalized to all $C^{1,1}$ knots and links by the author in [8].

For an arbitrary embedding of a knot, $FocRad(K) = (\sup \kappa)^{-1}$ may be small due to large (local) curvature κ in a small part of the curve. Y. Diao, C. Ernst and E. J. Janse Van Rensburg studied two different families of thicknesses, T_ε and t_ε generalizing $T_0 = t_0 = IR$ in [4]. Their main idea was to generalize the notions of the normal injectivity radius and the global radius of curvature by excluding certain open neighborhoods of the diagonal of $K \times K$. This allows the possibility of the intersections of the normal discs at points which are very close to each other along K, in T_ε for $\varepsilon > 0$, and reducing the effects of high curvature along small parts of a curve on its thickness. "These radii capture a balanced view between the geometric and the topological properties of these curves" [4, Abstract]. Our approach and results are quite different from [4]. We study a nonuniform thickness functional which allows a nonuniform distribution of the strength of forces along a curve in the Euclidean space. This model can help us to understand the differences in the shape (curvature) of a large nonuniform polymer at various points, when it is in an ideal configuration. We study the focal points, their relation to the local shape and the weight function, and the intersection behavior of the normal (spherical) discs close to the main diagonal in contrast to [4]. As a consequence, we are able to obtain results of geometric rigidity such as Theorem 2. Another point of difference is that we generalize the notion of thickness by allowing smooth variance instead of truncation. Although our proofs are of different in nature from [4], one unexpected observation is that Horizontal Collapsing Property of Theorem 2 implies that (our) almost injectivity radius AIR, and the maximum thickness t_m of [4] satisfy that $AIR \leq t_m$, provided that one adjusts t_m for the nonuniform setting.

In this article, we study a ball-model to describe nonuniform thickness. Most of the results of this article are true for surfaces or submanifolds of $\mathbf{R}^n$, but the results about the focal points are qualitative and the proofs are detailed. In order to have explicit expressions for the behavior and location of the singular (focal) points, and to be able to obtain the rigidity in Theorem 2, we concentrate on the curves in the Euclidean space. Even though our motivation comes from examples in $\mathbf{R}^3$, all results are stated and proved in $\mathbf{R}^n$ since our proofs are independent of the dimension of the ambient space, and they do not simplify for $n = 2$, 3. In our

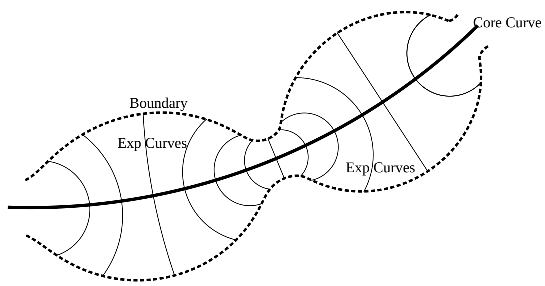

FIGURE 2.

model, a curve K is a union of finitely many disjoint closed curves and it is furnished with a weight function $\mu : K \to (0, \infty)$. The nonuniform $R-$tubular neighborhood $O(K, \mu R)$ is the union of metric balls of radius $R\mu(q)$ centered at each $q \in K$. As R increases, the size of these balls increase at fixed rate at each point, but the rate differs from point to point of K. This model is different from the disc-model which allows the growth of the normal discs at different rates. One of the reasons that we chose to investigate the ball-model is that the physical forces, such as electrical and magnetic forces have effects in every direction rather than being restricted to chosen planes. Furthermore, the ball-model can be investigated more thoroughly, since there is a natural potential function, $\min_{q \in K} \frac{\|p-q\|}{\mu(q)}$.

We study the problem by using distance function methods from Riemannian geometry. Throughout the article, we use the squared $\mu-$distance functions $\|p - x\|^2 \mu(x)^{-2}$. We define the generalized exponential function $\exp^\mu(q, Rv) = p$ to insure that q is a critical point of the restriction of $\|p - x\|^2 \mu(x)^{-2}$ to K. The image $\exp^\mu(NK_q)$ is going to be a sphere normal to K at q (with radius depending on μ where $\mu' \neq 0$) or a plane (only where $\mu' = 0$) normal to K at q, where NK_q denotes the set of vectors normal to K at q.

Even though there are many parallel results to the standard case ($\mu \equiv 1$), we also observed many contrasting cases which never occur in the standard case. In the standard case, the focal points occur at points $p = \exp(q, Rv)$ where the first and the second derivatives of the restriction of $E_p(x) = \|p - x\|^2$ to K are zero at q. The second derivatives become negative immediately after the focal points as R increases. Therefore, a line normal to K is never minimizing the distance to K past a focal point, and the exponential map can not be injective past a focal point. This is not always the case for nonconstant μ. First of all, $\exp^\mu(q, Rv)$ is not always a line for a fixed point q and a normal vector v. Since there is a quadratic term $\frac{R^2}{2}(\mu^2)''$ in the second derivative of the restriction of $F_p(x) = \|p - x\|^2 \mu(x)^{-2}$ to K, points p with zero second derivatives can be isolated away from the set of points with negative second derivatives, see Proposition 2. As a result, there are some cases with an exponential map which is a homeomorphism within the injectivity radius but not a diffeomorphism. In other words, the injectivity radius can be strictly larger than the $\mu-$distance to first focal points. As a consequence, we need to modify the notion of injectivity radius.

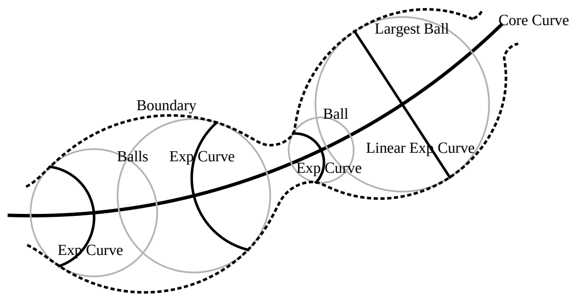

FIGURE 3. Some curves of type $\exp^{\mu}(\gamma(s_i), tN(s_i))$ for $-r < t < r$ and for some choices s_i are shown in the balls of radius $r\mu(s_i)$ and with center $\gamma(s_i)$, where N is the normal of $\gamma \subset \mathbf{R}^2$. Note the bending direction and the curvature of the exponential curves in the balls of radius μr.

DEFINITION 1. *Let K be a union of finitely many disjoint simple smoothly closed curves in $\mathbf{R}^n$, $\mu : K \to (0, \infty)$ be a C^2 function, and $grad\mu(q)$ be the gradient of μ. Let NK be the normal bundle of K in $\mathbf{R}^n$.*

Define $\exp^{\mu} : W \to \mathbf{R}^n$ *by*

$$\exp^{\mu}(q, w) = q - \mu(q) \|w\|^2 \, grad\mu(q) + \mu(q)\sqrt{1 - \|grad\mu(q)\|^2 \|w\|^2} w$$

where $W = \{w \in NK_q : q \in K \text{ and } \|w\| \leq \dfrac{1}{\|grad\mu(q)\|} \text{ when } \|grad\mu(q)\| \neq 0\}$.

Let γ be a parametrization of K locally with respect to arclength s. We use a standard abuse of notation $\mu(s) = \mu(\gamma(s))$. We can take the (intrinsic) gradient $grad\mu(\gamma(s)) = \mu'(s)\gamma'(s)$, since μ is defined only on K which is one dimensional, see Definition 6 and Remark 1 for justifications. Hence, we can rewrite $\exp^{\mu}$ as follows.

$$\exp^{\mu}(\gamma(s), w) = \gamma(s) - \mu(s)\mu'(s)\gamma'(s) \|w\|^2 + \mu(s)\sqrt{1 - (\mu'(s) \|w\|)^2} w$$

DEFINITION 2. *Let $D(r) = \{(q, w) \in NK : q \in K \text{ and } \|w\| < r\}$.*

i. The differentiable injectivity radius $DIR(K, \mu)$ is

$$\sup\{r : \exp^{\mu} \text{ restricted to } D(r) \text{ is a diffeomorphism onto its image}\}$$

ii. The topological injectivity radius $TIR(K, \mu)$ is

$$\sup\{r : \exp^{\mu} \text{ restricted to } D(r) \text{ is a homeomorphism onto its image}\}$$

iii. The almost injectivity radius $AIR(K, \mu)$ is

$$\sup \left\{ \begin{array}{c} r : \exp^{\mu} : U(r) \to U_0(r) \text{ is a homeomorphism where } U(r) \text{ is an open} \\ \text{and dense subset of } D(r), \text{ and } U_0(r) \text{ is an open subset of } \mathbf{R}^n. \end{array} \right\}$$

Observe that $r < TIR(K, \mu)$ is equivalent to that for all $p \in O(K, \mu r)$ there exists a unique minimum of $\|p - x\|^2 \mu(x)^{-2} : K \to \mathbf{R}$, i. e. there is a unique $\mu-$closest point of K to p. There are examples in $\mathbf{R}^n$ showing that $DIR(K, \mu) < TIR(K, \mu)$ and $TIR(K, \mu) < AIR(K, \mu)$ in every dimension $n \geq 2$, see section

5. In the $\mu = 1$ case, the injectivity radius functional is upper semicontinuous in the C^1 topology. As a consequence, thickest/tight/ideal knots and links exist for $\mu = 1$, see [2], [7], [8], [10], and [16]. There are examples in $\mathbf{R}^n$ with nonconstant μ showing that $DIR(K, \mu)$ and $TIR(K, \mu)$ are not upper semicontinuous, see Section 5. Hence, the existence of thickest/tight/ideal knots and links in DIR (or TIR) sense is not guaranteed in general.

In order to study the different notions of injectivity radius for the nonuniform (K, μ), we generalize the notion of double critical self distance, introduce two levels for the focal radius, $FocRad^0(K, \mu)$ and $FocRad^{-}(K, \mu)$, and the upper and lower radii $LR(K, \mu)$ and $UR(K, \mu)$. These definitions are given immediately after Theorem 1. $FocRad^{-}$ and $FocRad^0$ are not necessarily equal in general, due to certain "even" multiplicity zeroes of $\mu'' + \frac{1}{4}\kappa^2\mu = 0$. This difference allows interesting examples mentioned above, which do not occur in the $\mu = 1$ case.

Theorem 1. *Let K be a union of finitely many disjoint simple smoothly closed (possibly linked or knotted) curves in $\mathbf{R}^n$, and $\mu : K \to (0, \infty)$ be a C^2 function. Then, one has the following:*

i. $LR(K, \mu) = DIR(K, \mu) \leq TIR(K, \mu) \leq AIR(K, \mu) = UR(K, \mu)$.

ii. For a fixed choice of embedding $K \subset \mathbf{R}^n$, $LR(K, \mu) = UR(K, \mu)$ holds for μ in an open and dense subset of $C^3(K, (0, \infty))$ in the C^3- topology.

iii. Let $\{(K_i, \mu_i) : i = 1, 2, ...\}$ be a sequence where each K_i is a disjoint union of finitely many simple smoothly closed curves in $\mathbf{R}^n$ with C^2 weight functions, and similarly for (K_0, μ_0). If $(K_i, \mu_i) \to (K_0, \mu_0)$ in C^2 topology, then

$$\limsup_{i\to\infty} AIR(K_i, \mu_i) \leq AIR(K_0, \mu_0).$$

Definition 3. *A pair of points $(q_1, q_2) \in K \times K$ is called a double critical pair for (K, μ), if $q_1 \neq q_2$ and $grad\Sigma(q_1, q_2) = 0$, where $\Sigma : K \times K \to \mathbf{R}$ is defined by $\Sigma(q_1, q_2) = \|q_1 - q_2\|^2 (\mu(q_1) + \mu(q_2))^{-2}$.*

By taking parametrizations γ_1, γ_2 of K locally with respect to arclength s, and $\sigma(s, t) = \|\gamma_1(s) - \gamma_2(t)\|^2 (\mu(\gamma_1(s)) + \mu(\gamma_2(t))^{-2}$: (See Definition 6.)

$$grad\Sigma(q_1, q_2) = 0 \Leftrightarrow \nabla\sigma(s_1, s_2) = 0, \text{ where } q_i = \gamma_i(s_i) \text{ for } i = 1, 2.$$

Double critical self $\mu-$distance of (K, μ) is defined as

$$\frac{1}{2}DCSD(K, \mu) = \min\left\{ \frac{\|q_1 - q_2\|}{\mu(q_1) + \mu(q_2)} : (q_1, q_2) \text{ is a double critical pair for } (K, \mu) \right\}.$$

Definition 4. *If K is connected, by using a unit speed parametrization $\gamma(s)$: $\mathbf{R} \to K$, such that $\gamma(s + L) = \gamma(s)$ where L is the length of K, $\mu(s) = \mu(\gamma(s))$, and the curvature $\kappa(s)$ of $\gamma(s)$, one defines $FocRad^0(K, \mu)$ to be*

$$\left(\max \left[\begin{array}{c} \max \left\{ \begin{array}{c} \frac{1}{2}(\mu^2)'' + \frac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu\left(\mu'' + \frac{1}{4}\kappa^2\mu\right)} : \\ \text{where } \mu'' + \frac{1}{4}\kappa^2\mu \geq \mathbf{0} \end{array} \right\}, \\ \max\left\{ |\mu'|^2 : s \in Domain(\gamma) \right\} \end{array} \right] \right)^{-\frac{1}{2}}.$$

$FocRad^{-}(K, \mu)$ is defined similarly by using the following expression instead:

$$\left(\max \left[\begin{array}{c} \sup \left\{ \begin{array}{c} \frac{1}{2}(\mu^2)'' + \frac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\mu\left(\mu'' + \frac{1}{4}\kappa^2\mu\right)} : \\ \text{where } \mu'' + \frac{1}{4}\kappa^2\mu > \mathbf{0} \end{array} \right\}, \\ \max\left\{ |\mu'|^2 : s \in Domain(\gamma) \right\} \end{array} \right] \right)^{-\frac{1}{2}}.$$

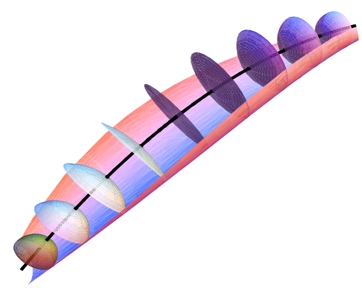

FIGURE 4. A 3-dimensional version of Figure 2. This shows some spherical caps of type $\exp^\mu(NK_q \cap D(r))$ normal to K, in the μr-neighborhood, for some choices of q on K. See Proposition 1.

If K has several components K_i, $i = 1, 2, ..., i_0$, then $FocRad^0(K, \mu)$ is the minimum of $FocRad^0(K_i, \mu)$ for $i = 1, 2, ...i_0$, and $FocRad^-(K, \mu)$ is the minimum of $FocRad^-(K_i, \mu)$ for $i = 1, 2, ...i_0$. The lower and upper radii are defined as follows:

$$LR(K, \mu) = \min\left(\frac{1}{2}DCSD(K, \mu), FocRad^0(K, \mu)\right)$$

$$UR(K, \mu) = \min\left(\frac{1}{2}DCSD(K, \mu), FocRad^-(K, \mu)\right).$$

If $\mu = 1$, then $FocRad^0(K, 1) = FocRad^-(K, 1) = (\max \kappa)^{-1}$. Lemma 2 provides us the characterization of $DCSD$ in terms of the angles that the line segment $\overline{q_1 q_2}$ makes with K at q_1 and q_2, generalizing the usual definition of $DCSD$ of the standard case where $\mu = 1$ and line segment $\overline{q_1 q_2}$ is perpendicular to K at both q_1 and q_2.

We studied the properties of the singular $\exp^\mu$ maps within $UR(K, \mu)$. Theorem 2 classifies all collapsing type singularities. If the injectivity of $\exp^\mu$ fails within $UR(K, \mu)$ radius, that is if two distinct points of $D(UR(K, \mu))$ are identified by $\exp^\mu$, then a curve of constant height in $D(UR(K, \mu))$ joining the identified points collapses to the same point under $\exp^\mu$. Figure 5 shows the unique way the injectivity of $\exp^\mu$ fails within $UR(K, \mu)$, up to rescaling and isometries of $\mathbf{R}^3$.

THEOREM 2. ***Horizontal Collapsing Property***

Let K be a union of finitely many disjoint simple smoothly closed (possibly linked or knotted) curves in $\mathbf{R}^n$, and $\mu : K \to (0, \infty)$ be a C^2 function. Assume that $\exp^\mu(q_1, r_1 v_1) = \exp^\mu(q_2, r_2 v_2) = p_0$ for r_1, $r_2 < UR(K, \mu)$, $v_i \in UNK_{q_i}$ with $(q_1, r_1 v_1) \neq (q_2, r_2 v_2)$. Then, one has the following:

i. q_1 and q_2 belong to the same component of K, which is denoted by K_1.

ii. Let $\gamma(s) : \mathbf{R} \to K_1 \subset \mathbf{R}^n$ be a unit speed parametrization of K_1 such that $\gamma(s + L) = \gamma(s)$ where L is the length of K_1, $N_\gamma(s)$ denotes the principal normal of γ, and $q_i = \gamma(s_i)$ for $i = 1, 2$ with $0 \leq s_1 < s_2 < L$. Then, $r_1 = r_2$, $v_i = N_\gamma(s_i)$ for $i = 1, 2$, and $\exp^\mu(\gamma(s), r_1 N_\gamma(s)) = p_0$, $\forall s \in I$ where $I = [s_1, s_2]$ or $[s_2 - L, s_1]$.

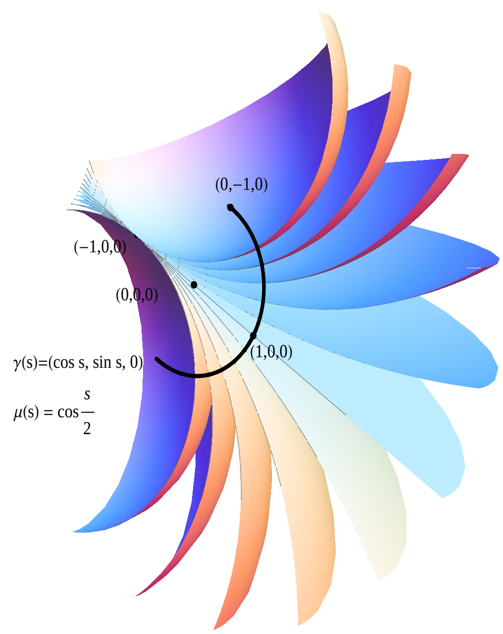

FIGURE 5. The normal exponential map from a portion of a unit circle with $\mu = \cos\frac{s}{2}$ in $\mathbf{R}^3$, showings some spherical caps of type $\exp^{\mu}(NK_q \cap D(r))$ normal to K. See Example 1B and Theorem 2.

iii. On the interval I, κ is a positive constant and all of the following hold:

$$(\mu')^2 - \mu\mu'' = \frac{1}{r_1^2} \text{ and } \gamma''' + \kappa^2\gamma' = 0,$$
$$\mu = \frac{2}{\kappa r_1}\cos\left(\frac{\kappa s}{2} + a\right) \text{ for some } a \in \mathbf{R}.$$

Therefore, Horizontal Collapsing occurs in a unique way only above arcs of circles of curvature κ and with specific μ. $\gamma(I) \neq K_1$, even if I is chosen to be a maximal interval satisfying above.

As a consequence, we can obtain $TIR(K,\mu)$ in terms of μ,κ, and $\frac{1}{2}DCSD(K,\mu)$. Theorems 2 and 3 give us a complete understanding of the differences between DIR, TIR and AIR.

THEOREM 3. *Let K be a union of finitely many disjoint simple smoothly closed (possibly linked or knotted) curves in $\mathbf{R}^n$. Let $\gamma : Domain(\gamma) \to K$ parametrize K with unit speed and $\mu(s) = \mu(\gamma(s))$. If $TIR(K,\mu) < UR(K,\mu)$, then K contains a circular arc of curvature κ and positive length, along which $\mu = \frac{2}{\kappa r}\cos\left(\frac{\kappa s}{2} + a\right)$ for some $a \in \mathbf{R}$ and $r < UR(K,\mu)$. In this case, $TIR(K,\mu)$ is equal to the infimum of such r.*

If K has no such circular arc with a compatible μ, that is, the set

$$\left\{\begin{array}{c} s \in Domain(\gamma) : \left(\mu'' + \frac{1}{4}\kappa^2\mu\right)(s) = 0, \text{ and } \kappa'(s) = 0 \text{ with } \kappa(s) > 0, \text{ and} \\ \gamma'''(s) + \kappa^2(s)\gamma'(s) = 0 \text{ and } \left(\mu'\right)^2(s) - \mu\mu''(s) = \frac{1}{r^2} \in \mathbf{R} \text{ where } r < UR(K,\mu). \end{array}\right\}$$

has no interior, then $TIR(K,\mu) = AIR(K,\mu) = UR(K,\mu)$.

The following theorem summarizes the remaining results obtained in the course of proving the theorems above, the exact structure of the singular set of $\exp^\mu$ within $UR(K,\mu)$, as well as the structure of the set of regular points.

THEOREM 4. *Let K_i denote the components of K. Let $\gamma_i : domain(\gamma_i) \to K_i$ be an onto parametrization of the component K_i with unit speed and $\mu_i(s) = \mu(\gamma_i(s))$. Then, the singular set $Sng^{NK}(K,\mu)$ of $\exp^\mu$ within $D(UR(K,\mu)) \subset NK$ is a graph over a portion of K:*

$$Sng^{NK}(K,\mu) = \textstyle\bigcup_i Sng_i^{NK}(K,\mu) \text{ and}$$

$$Sng_i^{NK}(K,\mu) = \left\{\begin{array}{c} (\gamma_i(s), R_i(s)N_{\gamma_i}(s)) \in NK_i \text{ where} \\ s \in domain(\gamma_i),\ \kappa_i(s) > 0, \\ \left(\mu_i'' + \frac{1}{4}\kappa_i^2\mu_i\right)(s) = 0, \text{ and} \\ 0 < R_i(s) = \left(\left((\mu_i')^2 - \mu_i\mu_i''\right)(s)\right)^{-\frac{1}{2}} < UR(K,\mu) \end{array}\right\}$$

where κ_i and N_{γ_i} are the curvature and the principal normal of γ_i, respectively. $D(UR(K,\mu)) - Sng^{NK}(K,\mu)$ is connected in each component of NK, when $n \geq 2$. Let

$$\begin{aligned} Sng(K,\mu) &= \exp^\mu(Sng^{NK}(K,\mu)), \\ A_q &= \exp^\mu\left(NK_q \cap D(UR(K,\mu))\right), \text{ and} \\ A_q^* &= \exp^\mu\left(NK_q \cap D(UR(K,\mu)) - Sng^{NK}(K,\mu)\right). \end{aligned}$$

i. $O(K,\mu UR(K,\mu)) - Sng(K,\mu)$ has a codimension 1 foliation by A_q^*, which are (possibly punctured) spherical caps or discs.

ii. $\exp^\mu(D(UR(K,\mu)) - Sng^{NK}(K,\mu)) = O(K,\mu UR(K,\mu)) - Sng(K,\mu)$.

iii. If $A_{q_1} \cap A_{q_2} \neq \varnothing$ for $q_1 \neq q_2$ then q_1 and q_2 must belong to the same component of K, and A_{q_1} intersects A_{q_2} tangentially at exactly one point $p_0 = \exp^\mu(q_1, r_1v_1) = \exp^\mu(q_2, r_2v_2)$ where $(q_i, r_iv_i) \in Sng^{NK}(K,\mu)$, for $i = 1, 2$.

The remaining definitions and notation are given in Section 2. The first and second order analysis of the $\mu-$distance functions, and basic properties of $\exp^\mu$ are studied in Section 3. Section 4 contains the proofs involving DIR and TIR. Section 5 has several examples shoving the deviation from the standard $\mu = 1$ case. AIR and Horizontal Collapsing Property are studied in Section 6 after the examples which give the motivation for many proofs.

2. Further Notation and Definitions

We assume that K is a union of finitely many disjoint simple smoothly closed (possibly linked or knotted) curves in $\mathbf{R}^n$. Hence, K is a $1-$dimensional compact submanifold of $\mathbf{R}^n$, with finitely many components. All parametrizations $\gamma : I \to K$ are with respect to arclength s and C^3, unless it is indicated otherwise. All $\mu : K \to (0, \infty)$ are at least C^3. For some compactness arguments on a K, we may take $Domain(\gamma)$ to be a disjoint union of $\mathbf{R}$ / $Length(K_i)\mathbf{Z}$ by considering γ as periodic function of period $length(K_i)$ on each component K_i.

Notation 1. *TK and NK denote the tangent and normal bundles of K in $\mathbf{R}^n$, respectively. UTK and UNK denote the unit vectors, NK_q denotes the set normal vectors to K at q, and similarly for the others. For $v \in T\mathbf{R}^n_q = TK_q \oplus NK_q$, v^T and v^N denote the tangential and normal components of v to K, respectively. $D(r)$ denotes $\{(q, w) \in NK : q \in K \text{ and } \|w\| < r\}$.*

Notation 2. *i. We use the standard distance function $d(p, q) = \|p - q\|$ in $\mathbf{R}^n$. $B(p, r)$ and $\bar{B}(p, r)$ denote open and closed metric balls. For $A \subset \mathbf{R}^n$, $B(A, r) = \{x \in X : d(x, A) < r\}$.*

ii. The unit direction vector from q to p is $u(q, p) = \frac{p-q}{\|p-q\|}$ for $p \neq q$.

Definition 5. *Let $K \subset \mathbf{R}^n$ and $\mu : K \to (0, \infty)$ be given. We define:*

i. The μR neighborhood of K, $O(K, \mu R) = \bigcup_{q \in K} B(q, \mu(q)R)$,

ii. For $p \in \mathbf{R}^n$,

$E_p : K \to \mathbf{R}$ by $E_p(x) = \|p - x\|^2$,

$F_p : K \to \mathbf{R}$ by $F_p(x) = \|p - x\|^2 \mu(x)^{-2}$, the square of the $\mu-$distance function from p,

$F_p^c : K \to \mathbf{R}$ by $F_p^c(x) = \|p - x\|^2 (\mu(x) + c)^{-2}$,

$G : \mathbf{R}^n \to \mathbf{R}$ by $G(p) = \min_{x \in K} F_p(x)$ so that $O(K, \mu R) = G^{-1}([0, R^2))$, and

$\Sigma : K \times K \to \mathbf{R}$ by $\Sigma(x, y) = \|x - y\|^2 (\mu(x) + \mu(y))^{-2}$,

Notation 3. *For a local parametrization $\gamma : I \to K$ with respect to arclength s, we will identify $\mu(s) = \mu(\gamma(s))$, $F_p(s) = F_p(\gamma(s)) = \|p - \gamma(s)\|^2 \mu(\gamma(s))^{-2}$, and similarly for all functions above. We use $s \in \mathbf{R}$, and x or $q \in K$ to avoid ambiguity.*

Definition 6. *For a C^1 function $\mu : K \to (0, \infty)$, $\operatorname{grad} \mu$ denotes the intrinsic gradient field of μ, that is the unique vector field tangential to K such that for every tangent vector $v \in TK_q$, the directional derivative of μ at q in the direction v along K is $v \cdot (\operatorname{grad} \mu)(q)$. For every C^1 extension $\widetilde{\mu}$ of μ to an open subset of $\mathbf{R}^n$, containing q, one has $(\operatorname{grad} \mu)(q) = (\nabla \widetilde{\mu}(q))^T$ where ∇ denotes the usual gradient in $\mathbf{R}^n$ defined by using the partial derivatives in $\mathbf{R}^n$. See [17], p. 96. Since K is one dimensional, one has*

$$(\operatorname{grad} \mu)(\gamma(s)) = \mu'(\gamma(s))\gamma'(s) = \mu'(s)\gamma'(s)$$

for a parametrization γ with respect to arclength.

Remark 1. *The last line above is justified by the Chain Rule:*

$$\begin{aligned}\mu'(s) &= \frac{d}{ds}\mu(\gamma(s)) = \frac{d}{ds}\widetilde{\mu}(\gamma(s)) = \nabla\widetilde{\mu}(\gamma(s)) \cdot \gamma'(s) = (\nabla\widetilde{\mu}(\gamma(s)))^T \cdot \gamma'(s) \\ &= (\operatorname{grad} \mu)(\gamma(s)) \cdot \gamma'(s).\end{aligned}$$

Remark 2. *For a given parametrization γ of K with respect to arclength, $\mu''(s_0)$, $\gamma''(s_0)$, $(\mu'(s_0))^2$, $\|grad\mu(q)\|$ and $F_p''(s_0)$ are calculated at $q = \gamma(s_0)$ by using the given parametrization. However, all of these quantities depend only on K, μ and q, but not on the choice of the parametrization with respect to arclength. Observe that when one reverses the orientation of a parametrization, both μ' and γ' change signs at q. $\mathrm{grad}\,\mu(q)$ and $\|grad\mu(q)\|$ are both well-defined. Although the sign of $\mu'(q)$ is ambiguous, depending on the orientation of γ, we can use $|\mu'(q)| = \|grad\mu(q)\|$. If $grad\mu(q) = 0$, then $\|grad\mu(q)\|^{-1}$ is taken to be $+\infty$. The definitions given in Section 1, exponential map, focal radii, double critical self distance by using a parametrization, are independent of the choice of the parametrization.*

Notation 4. *For any function $f : X \to Y$ and $Z \subset X$, $f \mid Z$ is the restriction of f to Z.*

Definition 7. *Let $\gamma : I \to K \subset \mathbf{R}^n$, $\mu : K \to (0, \infty)$, $p \in \mathbf{R}^n$ and $q = \gamma(s_0) \in K$ be given.*
$q \in CP(p)$, if q is a critical point of $F_p(x)$, that is $F_p'(s_0) = 0$,
$q \in CP(p, +)$, if $F_p'(s_0) = 0$ and $F_p''(s_0) > 0$,
$q \in CP(p, 0)$, if $F_p'(s_0) = 0$ and $F_p''(s_0) = 0$,
$q \in CP(p, -)$, if $F_p'(s_0) = 0$ and $F_p''(s_0) < 0$.

Definition 8. *The radius of regularity is*

$$RegRad(K, \mu) = \sup\{r : \exp^\mu \text{ restricted to } D(r) \text{ is a non-singular } C^1 \text{ map}\}.$$

3. Basic Properties of exp^μ

Remark 3. *If $f(s) = \frac{E(s)}{g(s)}$, then by logarithmic differentiation $\frac{f'}{f} = \frac{E'}{E} - \frac{g'}{g}$. If $f'(s_0) = 0$, then $\frac{E'}{E}(s_0) = \frac{g'}{g}(s_0)$ and $\frac{f''}{f}(s_0) = \left(\frac{E''}{E} - \frac{g''}{g}\right)(s_0)$.*

Notation 5. *For $q \in K$ and $p \in \mathbf{R}^n - \{q\}$:*
$\alpha(q, p) = \measuredangle(grad\mu(q), u(q, p))$ when $grad\mu(q) \neq 0$, and
$\alpha(q, p) = \frac{\pi}{2}$ when $grad\mu(q) = 0$.

Lemma 1. *For $q \in K$ and $p \in \mathbf{R}^n - \{q\}$, and $c \in [0, \infty)$,*

$$q \text{ is a critical point of } F_p^c(x) \iff u(q,p)^T = -\frac{\|p - q\|\, grad\mu(q)}{\mu(q) + c}.$$

If q is a critical point of $F_p^c(x)$, then

$$\cos\alpha(q, p) = -\frac{\|p - q\|\, \|grad\mu(q)\|}{\mu(q) + c} \text{ and hence } \frac{\pi}{2} \leq \alpha(q, p) \leq \pi.$$

Proof. For a given $\gamma : I \to K$ with $q = \gamma(s_0)$, $v = \gamma'(s_0)$, and $E(s) = \|p - \gamma(s)\|^2$, one has $E'(s_0) = 2\,(p - \gamma(s_0)) \cdot (-\gamma'(s_0)) = 2\,(p - q) \cdot (-v)$. If q is a critical point of $F_p^c(x)$, then s_0 is a critical point of

$$F_p^c(\gamma(s)) = \|p - \gamma(s)\|^2\, (\mu(s) + c)^{-2} = E(s)(\mu(s) + c)^{-2}.$$

By Remark 3:

$$\begin{aligned}
\frac{2\,(p-q)\cdot(-v)}{\|p-q\|^2} &= \frac{E'}{E}(s_0) = \frac{\left((\mu(s))+c)^2\right)'}{(\mu(s))+c)^2}(s_0) = \frac{2\mu'(s_0)}{\mu(s_0)+c} \\
-2u(q,p)\cdot v &= \|p-q\|\,\frac{2\mu'(s_0)}{\mu(s_0)+c} = \|p-q\|\,\frac{2\mu'(s_0)v}{\mu(s_0)+c}\cdot v \\
u(q,p)\cdot v &= -\|p-q\|\,\frac{grad\mu(q)}{\mu(q)+c}\cdot v \\
u(q,p)^T &= -\frac{\|p-q\|\,grad\mu(q)}{\mu(q)+c}
\end{aligned}$$

This argument is reversible for the converse. The statement for $\cos\alpha$ is obvious when $grad\mu(q) = 0 = u(q,p)^T$. In the other case, we have the following.

$$\begin{aligned}
\|grad\mu(q)\|\cos\alpha(q,p) &= u(q,p)\cdot grad\mu(q) \\
&= -\|p-q\|\,\frac{grad\mu(q)}{\mu(q)+c}\cdot grad\mu(q) \\
&= -\frac{\|p-q\|\,\|grad\mu(q)\|^2}{\mu(q)+c}
\end{aligned}$$

□

Proposition 1. *i.* $p = \exp^\mu(q,w)$ *if and only if*

$$\begin{cases} q\in CP(p),\ \|p-q\| = \|w\|\,\mu(q) \text{ and one of the following holds} \\ 1.\ p=q \text{ and } w=0, \text{ or} \\ 2.\ p\neq q,\ 0<\|w\|<\|grad\mu(q)\|^{-1},\ u(q,p)^N\neq 0 \text{ and } w = \frac{\|p-q\|u(q,p)^N}{\mu(q)\|u(q,p)^N\|}, \text{ or} \\ 3.\ p\neq q,\ \|w\| = \|grad\mu(q)\|^{-1}<\infty,\ u(q,p)^N = 0 \text{ where} \\ \qquad \|grad\mu(q)\|\,w \text{ is an arbitrary unit vector in } UNK_q. \end{cases}$$

ii. If $p = \exp^\mu(q,Rv)$ *for a unit vector* v *and* $R>0$*, then*

$$F_p(q) = R^2 \text{ and } \cos\alpha(q,p) = -R\,\|grad\mu(q)\| = -\left\|u(q,p)^T\right\| \text{ and}$$

$$\exp^\mu(q,Rv) = \begin{cases} q+\mu(q)R\left(\cos\alpha(q,p)\frac{grad\mu(q)}{\|grad\mu(q)\|}+\sin\alpha(q,p)v\right) & \text{if } grad\mu(q)\neq 0 \\ q+\mu(q)Rv & \text{if } grad\mu(q)=0 \end{cases}$$

iii. $\exp^\mu : W\to\mathbf{R}^n$ *is an onto map, where*
$W = \{w\in NK_q : q\in K$ *and* $\|w\|\leq\|grad\mu(q)\|^{-1}$ *when* $\|grad\mu(q)\|\neq 0\}$.

iv. $\exp^\mu$ *is* C^1 *on the interior of* W *and the differential* $d(\exp^\mu)(q,\mathbf{0}) = \mu(q)Id$. *Consequently, there exists* $\varepsilon>0$*, such that* $\exp^\mu$ *is a diffeomorphism on* $\{w\in NK_q : q\in K$ *and* $\|w\|<\varepsilon\}$ *by the Inverse Function Theorem.*

v. If $grad\mu(q)=0$*, then* $exp^\mu(NK_q)$ *is an* $(n-1)-$*dimensional plane normal to* K *at* q*. If* $grad\mu(q)\neq 0$*, then* $exp^\mu(NK_q\cap W)$ *is an* $(n-1)-$*dimensional sphere normal to* K *at* q*, with the radius* $\frac{1}{2}\frac{\mu(q)}{\|grad\mu(q)\|}$ *and the center at* $q-\frac{1}{2}\frac{\mu(q)grad\mu(q)}{\|grad\mu(q)\|^2}$.

vi. If $grad\mu(q)\neq 0$*, then* $exp^\mu(NK_q\cap W)\cap K$ *has at least two distinct points. Consequently,* $TIR(K,\mu) < \frac{1}{\max_{q\in K}\|grad\mu(q)\|}$.

Proof. i. ($\Longrightarrow$:) Assume that $p=\exp^\mu(q,w)$ for some $w\in NK_q$.

$grad\mu(q) \in TK_q$ and $w \in NK_q$.

$$p - q = -\mu(q)\left\|w\right\|^2 grad\mu(q) + \mu(q)\sqrt{1 - \left\|grad\mu(q)\right\|^2 \left\|w\right\|^2}w$$
$$\left\|p - q\right\| = \mu(q)\left\|w\right\|$$

Hence, $p = q$ if and only if $w = 0$. In this case, $p = q \in CP(p)$ since it is the absolute minimum of F_p, and we obtain (1). Without loss of generality, assume that $p \neq q$ and $w \neq 0$ at this point.

$$u(q,p)^T = \left(\frac{p-q}{\left\|p-q\right\|}\right)^T = -\left\|w\right\| grad\mu(q) = -\frac{\left\|p-q\right\| grad\mu(q)}{\mu(q)}$$

By Lemma 1, we conclude that $q \in CP(p)$.

$$u(q,p)^N = \left(\frac{p-q}{\left\|p-q\right\|}\right)^N = \sqrt{1 - \left\|grad\mu(q)\right\|^2 \left\|w\right\|^2}\frac{w}{\left\|w\right\|}$$

(2) If $\left\|w\right\| < \left\|grad\mu(q)\right\|^{-1}$, then we conclude that $u(q,p)^N \neq 0$, and consequently,

$$\frac{w}{\left\|w\right\|} = \frac{u(q,p)^N}{\left\|u(q,p)^N\right\|}$$
$$w = \left\|w\right\| \frac{u(q,p)^N}{\left\|u(q,p)^N\right\|} = \frac{\left\|p-q\right\|}{\mu(q)} \frac{u(q,p)^N}{\left\|u(q,p)^N\right\|}.$$

(3) If $\left\|w\right\| = \left\|grad\mu(q)\right\|^{-1}$, then $u(q,p)^N = 0$ and $p - q = -\mu(q)\left\|w\right\|^2 grad\mu(q) = -\mu(q)\left\|grad\mu(q)\right\|^{-2} grad\mu(q)$ is independent of the direction of w.

($\Longleftarrow$:) For the converse, assume that q is a critical point of $F_p(x)$ for some $p \in \mathbf{R}^n$ and $\left\|p - q\right\| = R\mu(q)$ for some R.

If $R = 0$, then $p = q = \exp^\mu(q, 0)$, for the case (1).

Suppose that $R > 0$. By Lemma 1 for $c = 0$, one obtains that

$$u(q,p)^T = -\frac{\left\|p-q\right\| grad\mu(q)}{\mu(q)} = -Rgrad\mu(q)$$
$$\cos\alpha(q,p) = -R\left\|grad\mu(q)\right\| = -\left\|u(q,p)^T\right\| \geq -1$$
$$\sin\alpha(q,p) = \sqrt{1 - \left\|grad\mu(q)\right\|^2 R^2} = \left\|u(q,p)^N\right\|.$$

(2) If $\sin\alpha(q,p) > 0$, then one takes $w = R\frac{u(q,p)^N}{\left\|u(q,p)^N\right\|}$ so that $R = \left\|w\right\|$ and

$$\begin{aligned} p - q &= R\mu(q)u(q,p) = R\mu(q)\left(u(q,p)^T + u(q,p)^N\right) \\ &= -R^2\mu(q)grad\mu(q) + \mu(q)\left\|u(q,p)^N\right\| w \\ &= \exp^\mu(q,w) - q. \end{aligned}$$

(3) If $\sin\alpha(q,p) = 0$, then $\cos\alpha(q,p) = -1 = -R\left\|grad\mu(q)\right\|.$

$$u(q,p) = u(q,p)^T = -\frac{grad\mu(q)}{\left\|grad\mu(q)\right\|}$$
$$p = q + \left\|p-q\right\| u(p,q) = q - R\mu(q)\frac{grad\mu(q)}{\left\|grad\mu(q)\right\|} = q - R^2\mu(q)grad\mu(q)$$
$$p = \exp^\mu(q, Rv), \forall v \in UNK_q$$

ii. This follows the proof of (i).

iii. For every $p \in \mathbf{R}^n$, the continuous map $F_p : K \to \mathbf{R}$ must have a minimum on compact K, and hence it has a critical point $q \in K$. By the construction in (i), $p = \exp^\mu(q, w)$ for some $w \in NK_q$, and $\|w\| = R \leq \|grad\mu(q)\|^{-1}$.

iv. $\exp^\mu(q, w) = q - \mu(q) \|w\|^2 grad\mu(q) + \mu(q)\sqrt{1 - \|grad\mu(q)\|^2 \|w\|^2} w$ is C^1 except when $\|grad\mu(q)\| \|w\| = 1$. For a fixed $q \in K$, $v \in UNK_q$ and taking $w = Rv$,

$$\begin{aligned} &\frac{d}{dR} \exp^\mu(q, Rv)|_{R=0} \\ &= \frac{d}{dR}\left(q - \mu(q)R^2 grad\mu(q) + \mu(q)\sqrt{1 - \|\mathrm{grad}\,\mu(q)\|^2 R^2} vR\right)\Big|_{R=0} \\ &= \mu(q)v \end{aligned}$$

v. $exp^\mu(NK_q)$ is an $(n-1)-$dimensional is a plane normal to K at q when $grad\mu(q) = 0$ by the definition of $\exp^\mu$.

Assume that $grad\mu(q) \neq 0,$ and choose an arbitrary $v \in UNK_q$. For every $p = \exp^\mu(q, Rv)$, where $0 \leq R \leq \|grad\mu(q)\|^{-1}$,

$$\cos(\pi - \alpha(q,p)) = R\|grad\mu(q)\| = \frac{\|p-q\|}{\mu(q)}\|grad\mu(q)\|$$

$$\|p - q\| = \frac{\mu(q)}{\|grad\mu(q)\|}\cos(\pi - \alpha(q,p))$$

where $\mu(q)\|grad\mu(q)\|^{-1}$ does not depend on p. This is an equation of a semi-circle in the polar coordinates of the 2-plane passing through q and parallel to $grad\mu(q)$ and v, where q is the origin, θ is angle from $-grad\mu(q)\|grad\mu(q)\|^{-1}$ turning towards v, and $r = \|p - q\|$. The radius of the circle is $\frac{1}{2}\mu(q)\|grad\mu(q)\|^{-1}$, the center is at $q - \frac{1}{2}\mu(q) grad\mu(q)\|grad\mu(q)\|^{-2}$, and the circle is tangent to v at q. Since the center and the radius depend only on q and not on v, one concludes that $exp^\mu(NK_q \cap W)$ is an $(n-1)-$dimensional sphere normal to K at q.

vi. Intuitively, since K goes into $\exp^\mu(NK_q \cap W)$ (an $(n-1)-$dimensional plane sphere in $\mathbf{R}^n$) transversally at q, it has to come out of it somewhere else. By using the mod-2 intersection theory [12], page 77, the mod 2 intersection number of K and $\exp^\mu(NK_q \cap W)$ must be zero mod 2, since one can isotope two compact submanifolds away from each other in $\mathbf{R}^n$. Since $q \in \exp^\mu(NK_q \cap W)$, and the intersection of K and $\exp^\mu(NK_q \cap W)$ is transversal at q, the number of points in $K \cap \exp^\mu(NK_q \cap W)$ is more than 1. For another point $q' \in K \cap \exp^\mu(NK_q \cap W)$, and for every open neighborhood U of q' in K with $q \notin U$, $\exp^\mu(\{(y, w) \in NK : y \in U \text{ and } \|w\| < \varepsilon\})$ intersects $\exp^\mu(NK_q \cap W)$ along an open subset. The injectivity of $\exp^\mu$ must fail strictly before reaching q' and the antipodal point of q in $\exp^\mu(NK_q \cap W)$, that is when $R = \|grad\mu(q)\|^{-1}$. □

Corollary 1. *By the proof of Proposition 1 (iii), for every $p \in O(K, \mu R)$, there exists $q \in K$ and $v \in UNK_q$ such that $p = \exp^\mu(q, rv)$ for some $r = \sqrt{G(p)} < R$. Consequently, $\exp^\mu(D(R)) = O(K, \mu R) = G^{-1}([0, R^2))$, for all $R > 0$.*

Lemma 2. *i. (q_1, q_2) is a double critical pair for (K, μ) if and only if there exists $R > 0$ and p on the line segment joining q_1 and q_2 such that $\|p - q_i\| = R\mu(q_i)$ and $p = \exp^\mu(q_i, Rv_i)$ with $v_i \in UNK_{q_i}$ for $i = 1$ and 2. Consequently, (q_1, q_2) is a double critical pair for (K, μ) if and only if $q_1, q_2 \in CP(p)$ and $F_p(q_1) = F_p(q_2) > 0$.*

ii. If (q_1, q_2) *is a double critical pair for* (K, μ)*, then for* $i = 1$ *and* 2*,*

$$\cos\alpha(q_i, p) = -\frac{\|q_1 - q_2\| \, \|grad\mu(q_i)\|}{\mu(q_1) + \mu(q_2)} = \frac{\|p - q_i\| \, \|grad\mu(q_i)\|}{\mu(q_i)} = -R \, \|grad\mu(q_i)\| \, .$$

Proof. Assume that (q_1, q_2) is a double critical pair for (K, μ) and take $R = \frac{\|q_1 - q_2\|}{\mu(q_1)+\mu(q_2)}$. There exists a unique p on the line segment joining q_1 and q_2 such that $\|p - q_i\| = R\mu(q_i)$ for $i = 1$ and 2. Let q_2 be fixed. $grad\Sigma(x, q_2) \mid_{x=q_1} = 0$, that is, q_1 is a critical point of $\left(\frac{\|x-q_2\|}{\mu(x)+\mu(q_2)}\right)^2 = F_{q_2}^{\mu(q_2)}(x)$. By Lemma 1,

$$\begin{aligned} u(q_1, p)^T = u(q_1, q_2)^T &= -\frac{\|q_1 - q_2\| \, grad\mu(q_1)}{\mu(q_1) + \mu(q_2)} \\ &= -Rgrad\mu(q_1) = -\frac{\|q_1 - p\| \, grad\mu(q_1)}{\mu(q_1)} \end{aligned}$$

and consequently $q_1 \in CP(p)$. By Proposition 1, $p = \exp^\mu(q_1, Rv_1)$ for some $v_1 \in UNK_{q_1}$. The q_2 case is similar. This argument is reversible for the converse. The second statement of (i) and (ii) are straightforward by using Lemma 1. $\square$

Lemma 3. *Let* $A, B, C \in \mathbf{R}$ *with* $A, B \geq 0$*,* $f(t) = 1 - \frac{1}{2}Ct^2 - At\sqrt{1 - B^2t^2}$ *for* $t \in I$*, where* $I = [0, \frac{1}{B}]$ *if* $B > 0$*, and* $I = [0, \infty)$ *if* $B = 0$*.*

i. Equation (3.1) has no solution when $\frac{C}{2} + \frac{A^2}{4} - B^2 < 0$ *or* $A = C = 0$ *:*

$$1 - \frac{1}{2}Ct^2 - At\sqrt{1 - B^2t^2} = 0 \text{ for } t \in I. \tag{3.1}$$

Assume $A^2 + C^2 \neq 0$ *and* $\frac{C}{2} + \frac{A^2}{4} - B^2 \geq 0$ *for the rest of the lemma.*

ii. $\frac{C}{2} + \frac{A^2}{2} > 0$*, and* $\frac{C}{2} + \frac{A^2}{2} \geq A\sqrt{\frac{C}{2} + \frac{A^2}{4} - B^2}$*, where the equality occurs if and only if* $B = C = 0 < A$*.*

iii. Equation (3.1), $f(t) = 0$ *has at most 2 solutions on* I*, and they are in the form* t_0^+ *or* t_0^- *when they exist:*

$$t_0^\pm = \left(\frac{C}{2} + \frac{A^2}{2} \pm A\sqrt{\frac{C}{2} + \frac{A^2}{4} - B^2}\right)^{-\frac{1}{2}} .$$

Both t_0^+ *and* t_0^- *are the solutions of (3.1) unless* $B = C = 0$ *(*$t_0^- = \infty \notin \mathbf{R}$*).* $t_0^- = \frac{1}{B}$ *if and only if* $2B^2 = C \neq 0$*. Also,* $t_0^\pm = \frac{1}{B}$ *if and only if* $2B^2 = C \neq 0 = A$*.*

iv. $f'(t) = 0$ *has at most one solution on* $(0, \frac{1}{B})$*.*

v. If $B = C = 0 < A$*, then* $t_0^+ = \frac{1}{A}$ *is the only solution of (3.1),*
and $f(t) < 0 \Longleftrightarrow t_0^+ < t$*.*

vi. If $\frac{C}{2} + \frac{A^2}{4} - B^2 = 0$*, then* $t_0^+ = t_0^-$ *is the only solution of (3.1),*
and $f(t) > 0$*, for all* $t \neq t_0^+$*.*

vii. If $\frac{C}{2} + \frac{A^2}{4} - B^2 > 0$ *and* $B^2 + C^2 \neq 0$ *then both* $t_0^+ < t_0^-$ *are the solutions of (3.1), and* $f(t) < 0 \Longleftrightarrow t_0^+ < t < t_0^-$*.*

Proof. Squaring both sides of $1 - \frac{1}{2}Ct^2 = At\sqrt{1 - B^2t^2}$ gives a quadratic equation in t^2, and then solve for $u = 1/t^2$. For (iv), substitute $t = \frac{1}{B}\sin\theta$. The rest is elementary and long. $\square$

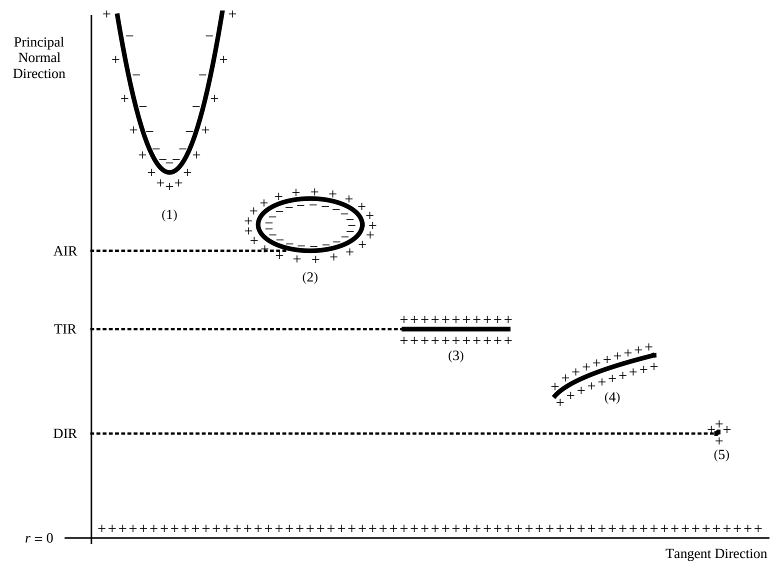

FIGURE 6. An example of the graph of the singular set in the domain of $\exp^\mu$ along the principal normal direction N of a curve γ of positive curvature is shown, as indicated in Proposition 2 and 5(ii). It is assumed that $DCSD$ is larger than $2FocRad^-$ in this example in order to indicate exact values of AIR, TIR, and DIR. The second derivative of the squared weighted distance function $\|p-x\|^2/\mu^2(x)$ is 0 along the singular set, and its signs at nearby points are indicated. Type (1) is the most common behavior, it is the only possibility when μ is sufficiently close to a constant, and it is the graph of $1/\kappa$ when $\mu = 1$. The "positive to negative and then back to positive" behavior shown in (2) occurs in Figure 8 (see Example 3), and Figure 11 (see Example 6). (3) depicts the Horizontal Collapsing Property, as in Figure 7 (see Example 1A) and Figure 5 (Example 1B). (5) is a "fake" focal point around which the μ-exponential map is a local homeomorphism but not a local diffeomorphism, as in Figure 10, (see Example 4).

PROPOSITION 2. *Let a local parametrization $\gamma : I \to K$ with respect to arclength s be given, $\kappa(s)$ denote the curvature of K at $\gamma(s)$, $\mu(s) = \mu(\gamma(s)) : I \to \mathbf{R}^+$, and $q = \gamma(s_0)$.*

i. If $p = \exp^\mu(q, Rv)$ for some $R \in (0, \|grad\mu(q)\|^{-1})$ and $v \in UNK_q$, then

$$F_p''(s_0) = \frac{2}{\mu^2(s_0)}\left(1 - \kappa(s_0)R\mu(s_0)\sqrt{1 - \|grad\mu(s_0)\|^2 R^2}\cos\beta - \frac{R^2}{2}(\mu^2)''(s_0)\right)$$

where $\beta = \measuredangle(\gamma''(s_0), u(q,p)^N)$ when both vectors are non-zero, and $\beta = 0$ otherwise.

ii. Let q and $v \in UNK_q$ be fixed, and R vary. For $p(R) = \exp^\mu(q, Rv)$, the sign of $\frac{d^2}{ds^2} F_{p(R)}(s)\big|_{s=s_0}$ behaves in only one of the following four manners, and in all cases $q \in CP(q,+)$ at $R = 0$:

a. $\forall R$, $q \in CP(p(R),+)$

b. $\exists R_1 > 0$, *such that*

$$q \in \begin{cases} CP(p(R), +) & \textit{if } R \in (0, R_1) \\ CP(p(R), 0) & \textit{if } R = R_1 \\ CP(p(R), -) & \textit{if } R \in (R_1, \|grad\mu(q)\|^{-1}) \end{cases}$$

c. $\exists R_2 > R_1 > 0$ *such that*

$$q \in \begin{cases} CP(p(R), +) & \textit{if } R \in (0, R_1) \cup (R_2, \|grad\mu(q)\|^{-1}) \\ CP(p(R), 0) & \textit{if } R = R_1 \textit{ or } R_2 \\ CP(p(R), -) & \textit{if } R \in (R_1, R_2) \end{cases}$$

d. $\exists R_1 > 0$ *such that*

$$q \in \begin{cases} CP(p(R), +) & \textit{if } R \neq R_1 \\ CP(p(R), 0) & \textit{if } R = R_1 \end{cases} .$$

Proof. i. To simplify the calculations, set $E(s) = \|p - \gamma(s)\|^2$ so that $F_p(s) = E(s)\mu(s)^{-2}$. Since $p = \exp^\mu(q, Rv)$, we already know that $F_p'(s_0) = 0$ and $\|p - q\| = R\mu(q)$ by Proposition 1(i). $\gamma''(s_0) = \kappa(s_0)N_\gamma(s_0)$ where $\kappa(s)$ is the curvature of $\gamma(s)$ in the ambient space $\mathbf{R}^n$, and $N_\gamma(s)$ is the principal normal of $\gamma(s)$ when $\kappa(s) > 0$. When $\kappa(s) = 0$, we will write $\gamma''(s) = \kappa(s)N_\gamma(s) = 0$ although $N_\gamma(s)$ is not defined. Since s is the arclength, $\gamma''(s_0) \in NK_q$. Let $\beta = \measuredangle(\gamma''(s_0), u(q,p)^N)$ when both vectors are non-zero, otherwise take $\beta = 0$.

$$\begin{aligned} \gamma''(s_0) \cdot (p - q) &= \gamma''(s_0) \cdot u(q,p) \|p - q\| = \gamma''(s_0) \cdot u(q,p)^N \|p - q\| \\ &= \kappa(s_0) \cos\beta \left\|u(q,p)^N\right\| \|p - q\| \\ &= \kappa(s_0) \cos\beta \sqrt{1 - \|grad\mu(q)\|^2 R^2} R\mu(s_0) \end{aligned}$$

$$\begin{aligned} E'(s) &= 2(p - \gamma(s)) \cdot (-\gamma'(s)) \\ E''(s) &= 2\gamma'(s) \cdot \gamma'(s) + 2(p - \gamma(s)) \cdot (-\gamma''(s)) \\ E''(s_0) &= 2[1 - (p - q) \cdot \gamma''(s_0)] \end{aligned}$$

$$\begin{aligned} F_p''(s_0) &= F_p(s_0) \left(\frac{E''}{E} - \frac{(\mu^2)''}{\mu^2} \right)(s_0) \\ &= \frac{\|p - q\|^2}{\mu^2(s_0)} \left(\frac{2[1 - (p - q) \cdot \gamma''(s_0)]}{\|p - q\|^2} - \frac{(\mu^2)''}{\mu^2}(s_0) \right) \\ &= \frac{2}{\mu^2(s_0)} \left(1 - \gamma''(s_0) \cdot (p - q) - \frac{\|p - q\|^2}{2\mu^2(s_0)} (\mu^2)''(s_0) \right) \\ &= \frac{2}{\mu^2(s_0)} \left(1 - \gamma''(s_0) \cdot (p - q) - \frac{R^2}{2} (\mu^2)''(s_0) \right) \\ &= \frac{2}{\mu^2(s_0)} \left(1 - \kappa(s_0) R\mu(s_0) \sqrt{1 - \|grad\mu(s_0)\|^2 R^2} \cos\beta - \frac{R^2}{2} (\mu^2)''(s_0) \right) \end{aligned}$$

ii. Observe that $F_p''(s_0) > 0$ for small $R > 0$, and the expression for $F_p''(s_0)$ is continuous in R, and it has at most two roots by Lemma 3. $\square$

Definition 9. *For one variable functions $\mu \in C^2$, and $\kappa \in C^0$, define:*

$$\Delta(\kappa, \mu) = \frac{1}{2}(\mu^2)'' + \frac{1}{4}\kappa^2\mu^2 - (\mu')^2 = \mu\left(\mu'' + \frac{\kappa^2}{4}\mu\right)$$
$$\Lambda(\kappa, \mu) = \frac{1}{2}(\mu^2)'' + \frac{1}{2}\kappa^2\mu^2 + \kappa\mu\sqrt{\Delta(\kappa, \mu)}$$

Observe that $\Delta(\kappa, \mu) = \frac{C}{2} + \frac{A^2}{4} - B^2$ and $\Lambda(\kappa, \mu) = \frac{C}{2} + \frac{A^2}{2} + A\sqrt{\frac{C}{2} + \frac{A^2}{4} - B^2}$, if $A = \kappa\mu$, $B = |\mu'|$ and $C = (\mu^2)''$, see Lemma 3.

Proposition 3. *i. Let K be connected, with a given (onto) parametrization $\gamma : Domain(\gamma) \to K$, with respect to arclength s, $\kappa(s)$ denote the curvature of K at $\gamma(s)$, $\mu(s) = \mu(\gamma(s)) : Domain(\gamma) \to \mathbf{R}^+$, and $q = \gamma(s_0)$. If the set*

$$\left\{R \in \left[0, \|grad\mu(q)\|^{-1}\right) : \exists v \in UNK_q,\ p = \exp^\mu(q, Rv) \text{ and } F_p''(s_0) = 0\right\}$$

is not empty, then its infimum is $\Lambda(\kappa, \mu)(s_0)^{-\frac{1}{2}}$.

ii. $\left\{s \in Domain(\gamma) : \mu'' + \frac{\kappa^2}{4}\mu > 0\right\} \neq \varnothing$.

iii. Both $FocRad^0(K, \mu)$ and $FocRad^-(K, \mu) \in \mathbf{R}^+$ are positive (finite) real numbers.

iv. If K has more than one component, then all of the above hold for each component, and the zero-focal radius of the union is the minimum zero-focal radii of all components.

Proof. i. For fixed $q \in K$ and R, and varying $v \in UNK_q$, the expression for $F_p''(s_0)$ in Proposition 2 is minimal for $\beta = 0$. If $\kappa(s_0) > 0$, then the minimum occurs when $v_0 = N_\gamma(s_0)$, and $p_0 = \exp^\mu(q, Rv_0)$. If $\kappa(s_0) = 0$, then $F_p''(s_0)$ does not depend on $\cos\beta$. Hence, for all $v \in UNK_q$, and $p = \exp^\mu(q, Rv)$:

$$F_p''(s_0) \geq F_{p_0}''(s_0) = \frac{2}{\mu^2(s_0)}\left(1 - \kappa(s_0)R\mu(s_0)\sqrt{1 - \|grad\mu(s_0)\|^2 R^2} - \frac{R^2}{2}(\mu^2)''(s_0)\right)$$

Assume that there is a solution of $F_p''(s_0) = 0$ with $R \in \left[0, \|grad\mu(q)\|^{-1}\right)$. In Lemma 3, if the smaller positive solution t_0^+ exists, then t_0^+ decreases as $A = \kappa(s_0)\mu(s_0)\cos\beta$ increases to $\kappa(s_0)\mu(s_0)$. The smallest solution of R for $F_{p_0}''(s_0) = 0$ is $\Lambda(\kappa, \mu)(s_0)^{-\frac{1}{2}}$, by Definition 9 and Lemma 3.

ii-iii. Since K is compact, there exists $s_1 \in Domain(\gamma)$ so that $\mu''(s_1) > 0$ unless μ is constant. Also, there exists $s_2 \in Domain(\gamma)$ so that $\kappa_\gamma(s_2) > 0$, in the case of constant μ. Hence, there exists s_i (for either $i = 1$ or 2) such that $\Delta(\kappa, \mu)(s_i) = \mu\left(\mu'' + \frac{\kappa^2}{4}\mu\right)(s_i) > 0$. Hence $\{s \in Domain(\gamma) : \Delta(\kappa, \mu)(s) \geq 0\}$ is a non-empty compact subset of $Domain(\gamma)$, and the maximum of $\Lambda(\kappa, \mu)$ is attained. This maximum must be positive by Lemma 3(ii). Although $|\mu'(s)|^{-1} \geq \Lambda(\kappa, \mu)(s)$ where $\Delta(s) \geq 0$, it is possible that maximum of $|\mu'(s)|$ to occur where $\Delta(s) < 0$. The proof for $FocRad^-(K, \mu)$ is similar, since $\Lambda(\kappa, \mu)$ is bounded.

iv. This follows Definition 4. □

4. *DIR* and *TIR*

Lemma 4.i is a well known result for $\mu = 1$, see [6] or [3] for example.

Lemma 4. *(Recall that $F_p(x) = \|p-x\|^2 \mu(x)^{-2}$ and $G(p) = \min_{x\in K} F_p(x)$.)*

i. Given $p \in \mathbf{R}^n$ and $q \in K$ such that $G(p) = F_p(q) = R^2 > 0$ so that $p = \exp^\mu(q, Rv)$ where $v \in UN_q$. $\forall w \in UT\mathbf{R}^n_p$ such that $u(p,q)\cdot w > 0$, there exists $\eta > 0$ such that $\forall t \in (0,\eta)$, $G(p+tw) < R^2$.

ii. If G is differentiable at p, then $\nabla G(p) = c_1 u(q,p)$ for some $c_1 \geq \frac{2\|p-q\|}{\mu^2(q)} > 0$ and $\nabla\sqrt{G}(p) = c_2 u(q,p)$ for some $c_2 \geq \frac{1}{\mu(q)} > 0$.

Proof. Let $\measuredangle(u(p,q), w) = \theta < \frac{\pi}{2}$.

i. By a simple acute triangle argument in $\mathbf{R}^n$, for all small $t > 0$:

$$R^2 = G(p) = \frac{\|p-q\|^2}{\mu^2(q)} > \frac{\|p+tw-q\|^2}{\mu^2(q)} \geq \min_{x\in K} F_{p+tw}(x) = G(p+tw)$$

ii. $\forall w \in UT\mathbf{R}^n_p$ such that $u(p,q)\cdot w = \cos\theta > 0$, and for all small $t > 0$, (by the Law of Cosines)

$$G(p) - G(p+tw) \geq \frac{\|p-q\|^2}{\mu^2(q)} - \frac{\|p+tw-q\|^2}{\mu^2(q)} = \frac{2t\|p-q\|\cos\theta - t^2}{\mu^2(q)}$$

$$\mu^2(q)\left(-\nabla G(p)\right)\cdot w \geq 2\|p-q\|\cos\theta > 0$$

Therefore, $\nabla G(p)$ points in the direction of $u(q,p) = -u(p,q)$.

$$\|\nabla G(p)\| \geq \frac{2\|p-q\|}{\mu^2(q)}$$

$$\nabla\sqrt{G} = \frac{1}{2\sqrt{G}}\nabla G$$

$$\left\|\nabla\sqrt{G}\right\| \geq \frac{1}{\mu(q)}$$

□

$DIR(K,\mu) = \min\left(\frac{1}{2}DCSD(K,\mu), RegRad(K,\mu)\right)$ in Proposition 5, generalizes a proposition in [3, p. 95] or [6, p. 274], about the injectivity radius of the standard exponential map $\exp_p$ from a point p in a Riemannian manifold for $\mu = 1$ to our case of nonconstant μ in $\mathbf{R}^n$. Their proofs use the local invertibility of $\exp_p$ where it is non-singular. However, our proofs must follow an altered course. Geodesics are not minimizing past focal points in the $\mu = 1$ case where $DIR(K,1) = TIR(K,1)$. Hence, $\exp^1$ fails to be injective past first focal point(s). For general μ, we have examples with $RegRad(K,\mu) < TIR(K,\mu)$, that is $\exp^\mu$ is injective past some focal points (Example 4), and it is possible to have $DIR(K,\mu) = LR(K,\mu) < TIR(K,\mu) < UR(K,\mu)$ (Examples 2, 4 and 5). The approach of the proof of Proposition 4 about TIR is in essence similar to the proofs in [3, p. 95], or [6, p. 274]. However, we use the positivity of the second derivatives instead of regularity of the exponential map. We discuss the relation of singular points and zeroes of the second derivatives to understand the relation of DIR with TIR.

Proposition 4. *i. If $R = TIR(K,\mu)$, then either $R = \frac{1}{2}DCSD(K,\mu)$ or there exists $q \in K$ and $p \in \mathbf{R}^n$ such that $\|p-q\| = R\mu(q)$ and $q \in CP(p,0)$.*

ii. $LR(K,\mu) \leq TIR(K,\mu) \leq UR(K,\mu)$.

Proof. First, we prove the second inequality of (ii):

Claim 1. $TIR(K,\mu) \leq FocRad^-(K,\mu)$.

Suppose that $FocRad^-(K,\mu) < TIR(K,\mu)$. Then, there exists $p = \exp^\mu(q_1, v_1)$ such that $FocRad^-(K,\mu) < \|v_1\| < TIR(K,\mu)$ and $q_1 \in CP(p,-)$. $F_p''(s_1) < 0$ for $\gamma : I \to K \subset \mathbf{R}^n$ with $q_1 = \gamma(s_1) \in K$. F_p can not attain its minimum at q_1. Consequently, $\exists q_2 \in K - \{q_1\}$ such that $F_p(q_2) = G(p) = \min_{x\in K} F_p(x) < F_p(q_1) = \|v_1\|^2$ and $q_2 \in CP(p)$. By Proposition 1, $p = \exp^\mu(q_2, v_2)$ for some $v_2 \in NK_{q_2}$ such that $\|v_2\|^2 = F_p(q_2) < \|v_1\|^2 < TIR(K,\mu)^2$. This implies that $\exp^\mu$ restricted to $D(r)$ is not injective for all r with $\|v_1\| < r < TIR(K,\mu)$ which contradicts with the definition of TIR. This proves Claim 1.

By Lemma 2, if $\{q_1, q_2\}$ is a critical pair, then there exists p on the line segment joining q_1 and q_2 such that $\|p - q_i\| = R\mu(q_i)$ and $p = \exp^\mu(q_i, Rv_i)$ for some $v_i \in UNK_{q_i}$ for $i = 1$ and 2, and injectivity of $\exp^\mu$ fails on $D(R+\varepsilon), \forall \varepsilon > 0$. Hence,

$$TIR(K,\mu) \leq \min\left(\frac{1}{2}DCSD(K,\mu), FocRad^-(K,\mu)\right) = UR(K,\mu). \tag{4.1}$$

The rest of (ii) is proved after (i).

(i) Since, $d(\exp^\mu(q,v))_{v=0} = \mu(q)Id$, and K is compact, there exists $r_0 > 0$, such that exp^μ restricted to $D(r_0)$ is a diffeomorphism. Let $R = \sup\{r : exp^\mu$ restricted to $D(r)$ is injective$\}$. $\exp^\mu : D(R) \to O(K,\mu R)$ is injective, since $\exp^\mu(q_1, w_1) = \exp^\mu(q_2, w_2)$ with $\max(\|w_1\|, \|w_2\|) < R$ would imply that $\max(\|w_1\|, \|w_2\|) < r$ for some $r < R$. $\exp^\mu : \overline{D(r)} \to \overline{O(K,\mu r)}$ is a homeomorphism onto its image $\forall r < R$, since it is continuous and injective on a compact domain. The map $\exp^\mu : D(r) \to O(K,\mu r)$ is onto by Corollary 1, and an open map into $\mathbf{R}^n$, since $O(K,\mu r)$ is an open subset of $\mathbf{R}^n$, $\forall r < R$. Hence, $\exp^\mu : D(R) \to O(K,\mu R)$ is continuous, open and injective, and therefore a homeomorphism. It follows that $R = TIR(K,\mu)$. $\forall m \in \mathbf{N}^+$, injectivity of $\exp^\mu$ fails on $D(R+\frac{1}{m})$, and there exist distinct $(y_m, v_m), (z_m, w_m) \in D(R+\frac{1}{m})$ such that $\exp^\mu(y_m, v_m) = \exp^\mu(z_m, w_m) = x_m \in \mathbf{R}^n$, $\|v_m\| < R+\frac{1}{m}$ and $\|w_m\| < R+\frac{1}{m}$. If both $\|v_m\| < R$ and $\|w_m\| < R$ were true simultaneously, exp^μ restricted to $D(r)$ would not be injective for some $r < R$. So, we can assume that $\|v_m\| \geq R, \forall m$. By compactness, there exist convergent subsequences (use index j instead of m_j) $y_j \to y_0$, $v_j \to v_0 \in NK_{y_0} \cap W$, $z_j \to z_0$ and $w_j \to w_0 \in NK_{z_0} \cap W$ as $j \to \infty$, such that $\exp^\mu(y_0, v_0) = \exp^\mu(z_0, w_0) = p$.

$$\|v_0\| = \lim \|v_j\| = R \text{ and } \|w_0\| = \lim \|w_j\| \leq R$$

Suppose that $\|w_0\| < R$. We showed that $exp^\mu : D(R) \to O(K,\mu R)$ is a homeomorphism onto an open subset of $\mathbf{R}^n$. Observe that $\exp^\mu(y_0, tv_0)$ is a curve starting at y_0, going to p at the boundary of $\exp^\mu(D(R))$, and $p = \exp^\mu(z_0, w_0)$ which is an interior point of $\exp^\mu(D(R))$. This leads to a contradiction. Hence,

$$\|w_0\| = \|v_0\| = R.$$

Let $\gamma : Domain(\gamma) \to K$ be a parametrization with respect to arclength such that $y_0 = \gamma(s_0)$ and $z_0 = \gamma(t_0)$.

Case 1. If $y_0 \in CP(p,0)$ or $z_0 \in CP(p,0)$, then the proof of (i) is finished. We also have $FocRad^0(K,\mu) \leq TIR(K,\mu)$ in this case.

Case 2. If $y_0 \in CP(p,-)$, that is $F_p''(s_0) < 0$, then it would still be true that $F_{p'}''(s_0) < 0$ for $p' = \exp^\mu(y_0, (1-\varepsilon)v_0)$ for some $\varepsilon > 0$. This would imply that $FocRad^-(K,\mu) \leq (1-\varepsilon)R < R$ which contradicts Claim 1. Hence, $y_0 \notin CP(p,-)$ and $z_0 \notin CP(p,-)$.

Case 3. $y_0 = z_0 \in CP(p,+)$ and $v_0 = w_0$.

$$\exists\varepsilon_1 > 0 \text{ with } I_1 = [s_0 - \varepsilon_1, s_0 + \varepsilon_1] \text{ such that}$$
$$\forall x \in B(p,\varepsilon_1), \forall s \in I_1,\ F_x''(s) > 0.$$

$\exists\varepsilon_2 \in (0,\varepsilon_1)$ with $I_2 = [s_0 - \varepsilon_2, s_0 + \varepsilon_2] \subset I_1$ and $\exists\delta > 0$ such that

i. $\forall s \in I_2 - \{s_0\},\ F_p(s) > F_p(s_0) = R^2$ and

ii. $\forall s \in \partial I_2,\ F_p(s) \geq (R+\delta)^2$.

$$\exists j_0, \forall j \geq j_0,\ \|x_j - p\| < \min\left(\frac{\delta \min \mu}{3}, \varepsilon_1\right),\ y_j \in \gamma(I_2) \text{ and } z_j \in \gamma(I_2)$$

$$\forall s \in \partial I_2 \text{ and } \forall j \geq j_0 :$$
$$\|\gamma(s) - x_j\| \geq \|\gamma(s) - p\| - \|p - x_j\| \geq \mu(s)(R+\delta) - \frac{\delta \min \mu}{3} \geq \mu(s)(R + \frac{2\delta}{3})$$

hence, $F_{x_j}(s) \geq \left(R + \dfrac{2\delta}{3}\right)^2$

$$\forall j \geq j_0,$$
$$\|y_0 - x_j\| \leq \|y_0 - p\| + \|p - x_j\| \leq \mu(s_0)R + \frac{\delta \min \mu}{3} \leq \mu(s_0)\left(R + \frac{\delta}{3}\right)$$
$$F_{x_j}(s_0) \leq \left(R + \frac{\delta}{3}\right)^2$$

The minima of F_{x_j} restricted to I_2 are attained in the interior of $I_2, \forall j \geq j_0$. The function $F_{x_j}(s)$ has interior strict local minima at both y_j and z_j by the choice of ε_2. We chose $(y_j, v_j) \neq (z_j, w_j)$ initially. The case of $y_j = z_j$ with $v_j \neq w_j$ and $\exp^\mu(y_j, v_j) = \exp^\mu(z_j, w_j)$ implies that $\|v_j\| = \|w_j\| = \|\mathrm{grad}\,\mu(y_j)\|^{-1} > TIR(K,\mu)$ by Proposition 1(ii, vi). There exist $j_1 \geq j_0$ such that $\forall j \geq j_1,\ y_j \neq z_j$. For otherwise, one would obtain $R = \|v_0\| = \|w_0\| = \|\mathrm{grad}\,\mu(y_0)\|^{-1} > TIR(K,\mu)$ which is not the case. There must be a local maximum of $F_{x_j}(s)$ between y_j and z_j at an interior point of $\gamma(I_2)$, which contradicts with the choice of ε_1. Case 3 can not occur.

Case 4. $y_0 = z_0$ and $v_0 \neq w_0$. The injectivity of $exp^\mu \mid (NK_{y_0} \cap W)$ can only fail at $\|v_0\| = \|w_0\| = \|grad\mu(y_0)\|^{-1}$, Proposition 1(ii). However, $\|grad\mu(y_0)\|^{-1} > R = TIR(K,\mu)$ by Proposition 1(vi). Case 4 can not occur.

Case 5. $y_0 \neq z_0$ with $y_0 \in CP(p,+)$ and $z_0 \in CP(p,+)$. Recall $y_0 = \gamma(s_0)$ and $z_0 = \gamma(t_0)$.

Claim 2. $u(p, y_0) = -u(p, z_0)$.

There exists $\varepsilon_1 > \varepsilon_2 > 0$ and $\delta > 0$ (as in Case 3) with $I_i = [s_0 - \varepsilon_i, s_0 + \varepsilon_i]$ and $J_i = [t_0 - \varepsilon_i, t_0 + \varepsilon_i]$ for $i = 1, 2$ such that

i. $\gamma(I_1) \cap \gamma(J_1) = \varnothing$,

ii. $\forall x \in B(p,\varepsilon_1)$ and $\forall s \in I_1 \cup J_1$, $F_x''(s) > 0$,

iii. $\forall s \in I_2 - \{s_0\}$, $F_p(s) > F_p(s_0) = R^2$ and $\forall s \in J_2 - \{t_0\}$, $F_p(s) > F_p(t_0) = R^2$, and

iv. $\forall s \in \partial I_2$, $F_p(s) \geq (R+\delta)^2$ and $\forall s \in \partial J_2$, $F_p(s) \geq (R+\delta)^2$.

Suppose that $u(p,y_0) \neq -u(p,z_0)$. There exists $w \in UT\mathbf{R}^n_p$ with $u(p,y_0)\cdot w > 0$ and $u(p,z_0)\cdot w > 0$. As in the proof of Lemma 4, there exists $\eta \in (0, \delta \min \mu)$ such that the point $p_1 = p + \eta w$ satisfies that

$$0 < \|y_0 - p_1\| < \|y_0 - p\| = R\mu(y_0)$$
$$0 < \|z_0 - p_1\| < \|z_0 - p\| = R\mu(z_0)$$

$$\begin{aligned}
&\forall s \in \partial I_2, \\
\|\gamma(s) - p\| &\geq (R+\delta)\mu(s) \\
\|\gamma(s) - p_1\| &\geq \|\gamma(s) - p\| - \|p - p_1\| \\
&\geq (R+\delta)\mu(s) - \delta \min \mu \\
&\geq R\mu(s) \\
F_{p_1}(s) &\geq R^2 \\
F_{p_1}(s_0) &= \|y_0 - p_1\|^2 \mu(y_0)^{-2} < R^2
\end{aligned}$$

The minimum of F_{p_1} restricted to I_2 is attained at $q_1 = \gamma(s_0')$ with $s_0' \in interior(I_2)$ and $F_{p_1}(q_1) < R^2$. In fact, q_1 is unique (see the very end of Case 3). Similarly, there exists $q_2 = \gamma(t_0')$ with $t_0' \in interior(J_2)$ such that $F_{p_1}(q_2) = \min(F_{p_1} \mid J_2) < R^2$. Clearly, $q_1 \neq q_2$. $p_1 = \exp^\mu(q_1, R_1 u_1) = \exp^\mu(q_2, R_2 u_2)$, for some $u_i \in UNK_{q_i}$ and $R_i < R$, for $i = 1, 2$. This would imply that exp^μ is not injective on $D(r)$ for some $r < R = TIR(K,\mu)$, which contradicts the definition of TIR. This concludes the proof of Claim 2, $u(p,y_0) = -u(p,z_0)$.

We have three colinear points y_0, p, z_0, where y_0 and z_0 are both in $CP(p)$ and $R = \frac{\|p-y_0\|}{\mu(y_0)} = \frac{\|p-z_0\|}{\mu(z_0)}$. By Lemma 2, $\{y_0, z_0\}$ is a critical pair for (K,μ) and $R \geq \frac{1}{2} DCSD(K,\mu)$. By (4.1), $R = TIR(K,\mu) = \frac{1}{2} DCSD(K,\mu)$. This finishes all cases for (i).

ii. Summarizing all the cases, we have either $FocRad^0(K,\mu) \leq TIR(K,\mu)$ in Case 1, or $TIR(K,\mu) = \frac{1}{2} DCSD(K,\mu)$ in Case 5.

$$LR(K,\mu) = \min\left(\frac{1}{2} DCSD(K,\mu), FocRad^0(K,\mu)\right) \leq TIR(K,\mu).$$

□

Lemma 5. *Let $\gamma(s) : I \to K$ be a parametrization of K with respect to arclength, $v(s) : I \to UNK$ be C^1 with $v(s) \in UNK_{\gamma(s)}$ and $R \in \mathbf{R}^+$ be such that $(\gamma(s), Rv(s)) \in interior(W)$ for $|s - s_0| < \varepsilon$, $\eta(s) = \exp^\mu(\gamma(s), Rv(s))$, $q = \gamma(s_0)$ and $p = \eta(s_0)$. Then,*

$$\eta'(s_0) \cdot \gamma'(s_0) = \frac{\mu^2(s_0)}{2} \frac{d^2}{ds^2} F_p(\gamma(s))|_{s=s_0} = \frac{\mu^2(s_0)}{2} F_p''(s_0)$$

$$\eta'(s_0) \cdot (\eta(s_0) - c(s_0)) = \frac{\mu^3(s_0)}{4\mu'(s_0)} \frac{d^2}{ds^2} F_p(\gamma(s))|_{s=s_0} = \frac{\mu^3(s_0)}{4\mu'(s_0)} F_p''(s_0)$$

provided that in the second equality one has $\mu'(s) \neq 0$ and $c(s) = \gamma(s) - \frac{\mu(s)}{2\mu'(s)}\gamma'(s)$ to be the center of the $n-1$ dimensional sphere containing $\exp^\mu(NK_{\gamma(s)} \cap W)$.

Proof. By the definition of $\exp^\mu$ and $grad\ \mu$, and proof of Proposition 2(i):

$$\eta = \gamma - \mu\mu' R^2 \gamma' + \mu R \sqrt{1 - (\mu' R)^2} v$$

$$\eta \cdot \gamma' = \gamma \cdot \gamma' - \mu\mu' R^2 = \gamma \cdot \gamma' - \frac{1}{2} R^2 \left(\mu^2\right)'$$

$$\eta' \cdot \gamma' = (\eta \cdot \gamma')' - \eta \cdot \gamma''$$

$$\eta' \cdot \gamma' = 1 + (\gamma - \eta) \cdot \gamma'' - \frac{1}{2} R^2 \left(\mu^2\right)'' \tag{4.2}$$

$$\eta'(s_0) \cdot \gamma'(s_0) = 1 - (p - q) \cdot \gamma''(s_0) - \frac{1}{2} R^2 \left(\mu^2\right)''(s_0)$$

$$= \frac{\mu^2(s_0)}{2} F_p''(s_0) = \frac{\mu^2(s_0)}{2} \frac{d^2}{ds^2} F_p(\gamma(s))|_{s=s_0} \tag{4.3}$$

For the second part, assume that $\mu'(s) \neq 0$ locally.

$$\eta = \gamma - \mu\mu' R^2 \gamma' + \mu R \sqrt{1 - (\mu' R)^2} v$$

$$c = \gamma - \frac{\mu}{2\mu'} \gamma'$$

$$\eta' \cdot (\eta - c) = \eta' \cdot \gamma' \left(-\mu\mu' R^2 + \frac{\mu}{2\mu'}\right) + \eta' \cdot v \left(\mu R \sqrt{1 - (\mu' R)^2}\right) \tag{4.4}$$

By $v \cdot \gamma' = v \cdot v' = 0$, $\gamma' \cdot \gamma' = v \cdot v = 1$, and the proof of Proposition 2(i):

$$\eta' \cdot v = \left(\gamma - \mu\mu' R^2 \gamma' + \mu R \sqrt{1 - (\mu' R)^2} v\right)' \cdot v$$

$$\eta' \cdot v = -\mu\mu' R^2 \gamma'' \cdot v + \left(\mu R \sqrt{1 - (\mu' R)^2}\right)' \tag{4.5}$$

$$\left(\mu R \sqrt{1 - (\mu' R)^2}\right) \left(\mu R \sqrt{1 - (\mu' R)^2}\right)' = \frac{1}{2} \left(\mu^2 R^2 \left(1 - (\mu' R)^2\right)\right)'$$

$$= \mu\mu' R^2 - \left(\mu (\mu')^3 + \mu^2 \mu' \mu''\right) R^4 \tag{4.6}$$

By the proof of Proposition 1(i) and $\gamma''(s) \in NK_{\gamma(s)}$:

$$\gamma'' \cdot (\eta - \gamma) = \gamma'' \cdot u(\gamma, \eta) R\mu = \gamma'' \cdot u(\gamma, \eta)^N R\mu$$

$$\gamma'' \cdot (\eta - \gamma) = \gamma'' \cdot v \left\| u(\gamma, \eta)^N \right\| R\mu = \gamma'' \cdot v R\mu \sqrt{1 - (\mu' R)^2} \tag{4.7}$$

By combining (4.5), (4.6), (4.7) and using (4.2) in the last step:

$$\eta' \cdot v \left(\mu R \sqrt{1 - (\mu' R)^2}\right) =$$

$$= \mu R \sqrt{1 - (\mu' R)^2} \left(-\mu\mu' R^2 \gamma'' \cdot v + \left(\mu R \sqrt{1 - (\mu' R)^2}\right)'\right)$$

$$= -\mu\mu' R^2 \left(\mu R \sqrt{1 - (\mu' R)^2}\right) \gamma'' \cdot v + \mu\mu' R^2 - \left(\mu (\mu')^3 + \mu^2 \mu' \mu''\right) R^4$$

$$= -\mu\mu' R^2 \gamma'' \cdot (\eta - \gamma) + \mu\mu' R^2 - \mu\mu' \left((\mu')^2 + \mu\mu''\right) R^4$$

$$= \mu\mu' R^2 \left(1 - \gamma'' \cdot (\eta - \gamma) - \frac{1}{2} R^2 \left(\mu^2\right)''\right)$$

$$= \mu\mu' R^2 \left(\eta' \cdot \gamma'\right) \tag{4.8}$$

By combining (4.4), (4.8) and using (4.3) in the last step:

$$\begin{aligned}
\eta' \cdot (\eta - c) &= \left(-\mu\mu' R^2 + \frac{\mu}{2\mu'} \right) (\eta' \cdot \gamma') + \mu\mu' R^2 (\eta' \cdot \gamma') \\
&= \frac{\mu}{2\mu'} (\eta' \cdot \gamma') \\
\eta'(s_0) \cdot (\eta(s_0) - c(s_0)) &= \frac{\mu(s_0)}{2\mu'(s_0)} \eta'(s_0) \cdot \gamma'(s_0) = \frac{\mu(s_0)}{2\mu'(s_0)} \frac{\mu^2(s_0)}{2} F_p''(s_0) \\
&= \frac{\mu^3(s_0)}{4\mu'(s_0)} F_p''(s_0)
\end{aligned}$$

□

Proposition 5. *Let K be a union of finitely many disjoint simple smoothly closed curves in $\mathbf{R}^n$, and $\mu : K \to (0, \infty)$ be given.*

i. $\exp^\mu$ *restricted to the normal plane* $NK_q \cap int(W)$ *is non-singular, for each* $q \in K$. $\exp^\mu$ *is singular at the boundary of* W *where the spheres* $\exp^\mu(NK_q \cap W)$ *close up at the antipodal of* q.

ii. Let (q, w) *be an interior point of* W, $\exp^\mu(q, w) = p$, $\gamma : I \to K$ *be a parametrization of* K *with respect to arclength and* $q = \gamma(s_0)$.

$$\exp^\mu \text{ is singular at } (q, w) \text{ if and only if } \frac{d^2}{ds^2} F_p(\gamma(s))\big|_{s=s_0} = 0.$$

iii. (Recall Definition 8 of RegRad *in Section 2.)*

$$RegRad(K, \mu) = FocRad^0(K, \mu)$$

$$DIR(K, \mu) = LR(K, \mu) = \min\left(\frac{1}{2} DCSD(K, \mu), RegRad(K, \mu) \right)$$

Proof. i. For a fixed q, by Proposition 1(ii):

$$\exp^\mu(q, Rv) = \begin{cases} q + \mu(q) R \left(\cos\alpha(R) \frac{grad\mu(q)}{\|grad\mu(q)\|} + \sin\alpha(R) v \right) & \text{if } grad\mu(q) \neq 0 \\ q + \mu(q) R v & \text{if } grad\mu(q) = 0 \end{cases}$$

$$\text{where } \cos\alpha(R) = -R \|grad\mu(q)\| \text{ and } \sin\alpha(R) = \sqrt{1 - (R \|grad\mu(q)\|)^2}.$$

If $grad\mu(q) = 0$, $\exp^\mu$ restricted to NK_q is a dilation and translation, and it is non-singular along NK_q. If $grad\mu(q) \neq 0$, for each fixed $v \in UNK_q$, $\exp^\mu(q, Rv)$ follows the great circles of the sphere $\exp^\mu(NK_q \cap W)$ starting at q with non-zero speed until $q' = \exp^\mu(q, v \|grad\mu(q)\|^{-1})$ and $\exp^\mu$ is non-singular along $NK_q \cap int(W)$. However, $q' = \exp^\mu(q, v \|grad\mu(q)\|^{-1})$ for all $v \in UNK_q$, the sphere $\exp^\mu(NK_q \cap W)$ closes up at q', the antipodal of q. Hence, $\exp^\mu$ is singular along $NK_q \cap \partial W$.

ii. **Case 1**. $\mu'(s_0) \neq 0$.

Assume that $\exp^\mu$ is singular at (q, w) where $\exp^\mu(q, w) = p$, $(q, w) \in int(W)$. There exists a regular curve $\overline{\beta}(t)$ in NK, such that $\overline{\beta}(t_0) = (q, w)$ and $\exp^\mu(\overline{\beta}(t))$ is singular at $t = t_0$. $\overline{\beta}(t) = (\overline{\gamma}(t), \overline{R}(t)\overline{v}(t))$ for $\overline{v}(t) \in UNK_{\overline{\gamma}(t)}$. By (i), the singular directions can not be tangential to NK_q, and $0 \neq \frac{d\overline{\gamma}}{dt}(t_0) = \frac{d\overline{\gamma}}{ds}\frac{ds}{dt}(t_0)$. Hence, one can reparametrize $\overline{\beta}(t) = \beta(s) = (\gamma(s), R(s)v(s))$, with respect to the arclength s of γ for $|s - s_0| < \varepsilon$, and $s(t_0) = s_0$, and still have a regular curve $\beta(s)$ such that $\exp^\mu(\beta(s)) = \exp^\mu(\gamma(s), R(s)v(s))$ is singular at $s = s_0$. The curve $\varphi(R) =$

$\exp^\mu(\gamma(s_0), Rv(s_0))$ lies on the sphere $\exp^\mu(NK_q \cap W)$ with center $c(s_0)$ and it is normal to the radial vectors from the center. The curve $\eta(s) = \exp^\mu(\gamma(s), R(s_0)v(s))$ satisfies Lemma 5(ii), and $p = \eta(s_0) = \varphi(R(s_0))$.

$$
\begin{aligned}
0 &= \frac{d}{ds} \exp^\mu(\beta(s))|_{s=s_0} \\
&= \frac{d}{ds} \exp^\mu(\gamma(s), R(s_0)v(s))|_{s=s_0} + \left.\frac{dR}{ds}\right|_{s=s_0} \frac{d}{dR} \exp^\mu(\gamma(s_0), Rv(s_0))|_{R=R(s_0)} \\
0 &= \frac{d}{dR} \exp^\mu(\gamma(s_0), Rv(s_0))|_{R=R(s_0)} \cdot (\varphi(R(s_0)) - c(s_0)) \\
0 &= \frac{d}{ds} \exp^\mu(\gamma(s), R(s_0)v(s))|_{s=s_0} \cdot (\eta(s_0) - c(s_0)) \\
&= \frac{d\eta}{ds}(s_0) \cdot (\eta(s_0) - c(s_0)) = \frac{\mu^3(s_0)}{4\mu'(s_0)} F_p''(s_0)
\end{aligned}
$$

This finishes the proof of ($\Rightarrow$) in Case 1.

Assume that $F_p''(s_0) = 0$ where $\exp^\mu(q, w) = p$, and $(q, w) \in int(W)$. Consider $\eta(s) = \exp^\mu(\gamma(s), Rv(s))$ where $v(s) : I \to UNK$ be C^1 with $v(s) \in UNK_{\gamma(s)}$ and $R \in \mathbf{R}^+$ be such that $(\gamma(s), Rv(s)) \in interior(W)$ for $|s - s_0| < \varepsilon$, and $w = Rv(s_0)$.

$$
0 = \frac{\mu^3(s_0)}{4\mu'(s_0)} F_p''(s_0) = \eta'(s_0) \cdot (\eta(s_0) - c(s_0))
$$

The non-zero vector $(\gamma'(s_0), Rv'(s_0))$ is not tangential to $NK_q \cap int(W)$. $\eta'(s_0)$ is either zero or it is normal to the radial vector $\eta(s_0) - c(s_0)$. Therefore, $\eta'(s_0)$ is tangent to the $n-1$ dimensional sphere $\mathbf{S} = \exp^\mu(NK_q \cap W)$ at p.

$$
d(\exp^\mu)(q, w) : T(NK)_{(q,w)} = T(NK_q)_w \oplus \mathbf{R} \approx \mathbf{R}^n \to T\mathbf{R}^n_p = T\mathbf{S}_p \oplus \mathbf{R} \approx \mathbf{R}^n
$$

$$
\begin{aligned}
d(\exp^\mu)(q, w)|T(NK_q)_w : T(NK_q)_w &\to T\mathbf{S}_p \text{ is an isomorphism by (i)} \\
(\gamma'(s_0), Rv'(s_0)) &\in T(NK)_{(q,w)} \\
(\gamma'(s_0), Rv'(s_0)) &\notin T(NK_q)_w \\
d(\exp^\mu)(q, w)((\gamma'(s_0), Rv'(s_0))) = \eta'(s_0) &\in T\mathbf{S}_p
\end{aligned}
$$

$$
d(\exp^\mu)(q, w) : T(NK)_{(q,w)} \approx \mathbf{R}^n \to T\mathbf{R}^n_p \approx \mathbf{R}^n \text{ is not one-to one.}
$$

Therefore, $\exp^\mu$ is singular at (q, w) to conclude the proof of ($\Leftarrow$) in Case 1.

Case 2. $\mu'(s_0) = 0$. The proof is essentially the same as in Case 1 by replacing all "$\cdot\,(\eta(s_0) - c(s_0))$" with "$\cdot\gamma'(s_0)$" , since $\exp^\mu(NK_q)$ is an $n-1$ dimensional plane through $q = \gamma(s_0)$ normal to $\gamma'(s_0)$, and one uses the first equation of Lemma 5, $\eta'(s_0) \cdot \gamma'(s_0) = \frac{1}{2}\mu^2(s_0) F_p''(s_0)$ instead of the second equation.

iii. $RegRad(K, \mu) = FocRad^0(K, \mu)$ immediately follows (ii) and the definitions. Combining Proposition 4, definitions of $DIR(K, \mu)$, $TIR(K, \mu)$, $LR(K, \mu)$

and $UR(K,\mu)$:

$$\begin{aligned}
LR(K,\mu) &\le TIR(K,\mu) \le UR(K,\mu)\\
LR(K,\mu) &= \min\left(\frac{1}{2}DCSD(K,\mu), FocRad^0(K,\mu)\right)\\
UR(K,\mu) &= \min\left(\frac{1}{2}DCSD(K,\mu), FocRad^-(K,\mu)\right)\\
DIR(K,\mu) &\le TIR(K,\mu) \le \frac{1}{2}DCSD(K,\mu)\\
DIR(K,\mu) &\le RegRad(K,\mu) = FocRad^0(K,\mu)\\
DIR(K,\mu) &\le \min\left(\frac{1}{2}DCSD(K,\mu), RegRad(K,\mu)\right)
\end{aligned}$$

For all $0 < r < \min\left(\frac{1}{2}DCSD(K,\mu), RegRad(K,\mu)\right) \le TIR(K,\mu)$, $\exp^\mu$ restricted to $D(r)$ is a homeomorphism onto an open subset $O(K,\mu r)$ of $\mathbf{R}^n$ by the proof of Proposition 4(i), it is C^1 and non-singular, by Proposition 1. $\exp^\mu$ restricted to $D(r)$ is a diffeomorphism, for all $0 < r < \min\left(\frac{1}{2}DCSD(K,\mu), RegRad(K,\mu)\right)$, by the Inverse Function Theorem.

$$\begin{aligned}
DIR(K,\mu) &= \min\left(\frac{1}{2}DCSD(K,\mu), RegRad(K,\mu)\right)\\
&= \min\left(\frac{1}{2}DCSD(K,\mu), FocRad^0(K,\mu)\right) = LR(K,\mu)
\end{aligned}$$

□

Lemma 6. *$LR(K,\mu) = UR(K,\mu)$ holds for μ on an open and dense subset of $C^3(K,(0,\infty))$ in the C^3- topology, for a fixed choice of embedding $K \subset \mathbf{R}^n$.*

Proof. For simplicity, we will assume that K has one component. For a given onto parametrization $\gamma : domain(\gamma) = \mathbf{R}/(lengthK)\mathbf{Z} \to K$, that is given $\kappa(s)$, define $X_\kappa = \left\{\mu \in C^3(K,(0,\infty)) : 0 \text{ is a regular value of } \mu'' + \frac{\kappa^2}{4}\mu\right\}$. This condition is equivalent to "the graph of $\mu'' + \frac{\kappa^2}{4}\mu$ intersects $s-$axis transversally at every point of intersection" and it implies that $\left\{s : \left(\mu'' + \frac{1}{4}\kappa^2\mu\right)(s) = 0\right\}$ is a subset of the closure of $\left\{s : \left(\mu'' + \frac{1}{4}\kappa^2\mu\right)(s) < 0\right\}$ to conclude that $FocRad^0(K,\mu) = FocRad^-(K,\mu)$. X_κ is an open subset, since it is defined by an open condition, regularity. X_κ is dense in $C^3(K,(0,\infty))$, if we prove that for every given μ, we have $\mu_\varepsilon = \mu - \varepsilon\mu_0$ in X_κ for almost all small $|\varepsilon|$, for a fixed and appropriate choice of μ_0. κ can not be zero everywhere, since K is compact. Choose $\mu_1 : domain(\gamma) \to (0,\infty)$ such that $\mu_1''(s) > 0$ on a proper open subinterval of $domain(\gamma)$, containing the points where $\kappa(s) = 0$. Choose $c_1 > 0$ sufficiently large so that $\mu_0 = \mu_1 + c_1$ satisfies that $\mu_0'' + \frac{\kappa^2}{4}\mu_0 = \mu_1'' + \frac{\kappa^2}{4}\mu_1 + \frac{\kappa^2}{4}c_1 > 0$. Let $f = \left(\mu'' + \frac{1}{4}\kappa^2\mu\right)\left(\mu_0'' + \frac{1}{4}\kappa^2\mu_0\right)^{-1} : domain(\gamma) \to \mathbf{R}$. By Proposition 3(ii), $\mu'' + \frac{\kappa^2}{4}\mu \le 0, \forall s$ is not possible. If $\mu'' + \frac{\kappa^2}{4}\mu > 0, \forall s$, then $\mu \in X_\kappa$ which is open, and the proof is done. If $\mu'' + \frac{\kappa^2}{4}\mu > 0, \forall s$ is not true, then f is not constant, and $range(f) = [a,b]$ with $a \le 0 < b$. By Sard's Theorem [15], for almost all $\varepsilon \in range(f)$, ε is a regular value of f (that is $f(s) = \varepsilon$ and $f'(s) = 0$ have no common roots). Consequently, for the same ε, 0 is a regular value of

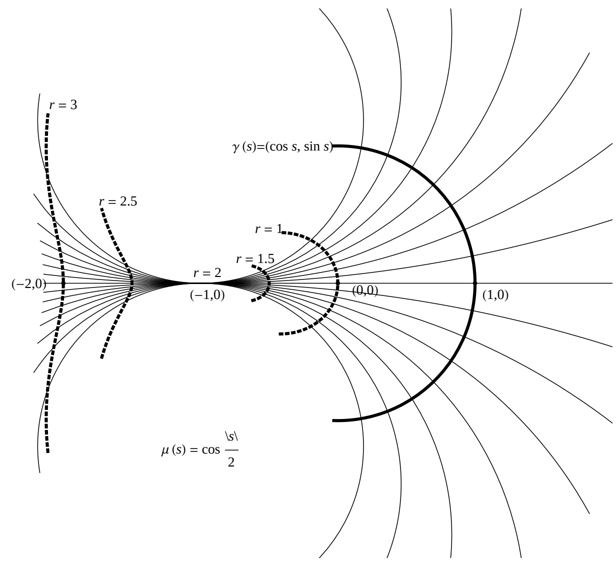

FIGURE 7. $\gamma(s) = (\cos s, \sin s)$ and $\mu(s) = \cos \frac{s}{2}$. This figure depicts the Horizontal Collapsing Property in dimension 2.

$\mu''_\varepsilon + \frac{1}{4}\kappa^2\mu_\varepsilon = \mu'' + \frac{1}{4}\kappa^2\mu - \varepsilon\left(\mu''_0 + \frac{1}{4}\kappa^2\mu_0\right)$. Hence, μ_ε is in X_κ for almost all small ε. $\square$

5. Examples

We will use the pointwise focal radii for $\gamma(s)$ and $\mu(s)$ in the examples:
$FocRad^0(\gamma(s), \mu(s)) = \Lambda(\kappa, \mu)(s)^{-\frac{1}{2}}$ if $\Delta(\kappa, \mu)(s) \geq 0$, and $|\mu'(s)|^{-1}$ otherwise.
$FocRad^-(\gamma(s), \mu(s)) = \Lambda(\kappa, \mu)(s)^{-\frac{1}{2}}$ if $\Delta(\kappa, \mu)(s) > 0$, and $|\mu'(s)|^{-1}$ otherwise.

EXAMPLE 1. *A. Figure 7. Let* $\gamma(s) = (\cos s, \sin s) : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to K \subset \mathbf{S}^1 \subset \mathbf{R}^2$ *and* $\mu(s) = \cos \frac{s}{2}$. *$K$ is the half of $\mathbf{S}^1$ with $x > 0$. For all s,*

$$\Delta(\kappa, \mu) = \mu\left(\mu'' + \frac{1}{4}\mu\right) = 0$$

$$\Lambda(\kappa, \mu) = \frac{1}{2}(\mu^2)'' + \frac{1}{2}\mu^2 = \frac{1}{4}$$

$$FocRad^0(K, \mu) = 2$$

$$FocRad^-(K, \mu) = \inf |\mu'(s)|^{-1} = \inf 2\left|\sin \frac{s}{2}\right|^{-1} = 2\sqrt{2}$$

$$FocRad^0(K, \mu) < FocRad^-(K, \mu)$$

Since $\mu'(0) = 0$, $\exp^\mu(NK_{(1,0)})$ is the $x - axis$. For $s \neq 0$, $exp^\mu(NK_{\gamma(s)} \cap W)$ is a circle of radius $\left|\frac{\mu}{2\mu'}\right| = \left|\cot \frac{s}{2}\right|$ and with center $\gamma - \frac{\gamma'\mu}{2\mu'} = (-1, \cot \frac{s}{2})$. For $s \neq 0$, all exp^μ-circles are tangent to $x - axis$ at $(-1, 0)$, and all intersecting $\mathbf{S}^1$ perpendicularly at both points of intersection. For all s, $\exp^\mu(\gamma(s), 2(-\cos s, -\sin s)) = (-1, 0)$. Hence, $\exp^\mu$ is singular and not injective along the $R = 2$ curve in NK. However, $\exp^\mu$ is still injective for $R > 2$. This type of singularity does not occur *for ($\mu = 1$)-exponential map in which case after the first focal point the exponential map is not injective.*

B. Figure 5. Let $\gamma(s) = (\cos s, \sin s, 0, ..., 0) : [a,b] \to K \subset E_{12} \subset \mathbf{R}^n$ *and* $\mu(s) = \cos\frac{s}{2}$, *where* E_{12} *is the* $2-$ *plane with* $x_i = 0$ *for* $i \geq 3$ *and* $[a,b] \subset (-\pi/2, \pi/2)$. $\exp^\mu(NK_{(1,0,..0)})$ *is the* $x_2 = 0$ *hyperplane, and all the spheres containing* $\exp^\mu(NK_q \cap W)$ *have centers on* E_{12} *and* $\exp^\mu(NK_q \cap W) \cap E_{12}$ *are the circles discussed in part A. Consequently, all* $\exp^\mu(NK_q \cap W)$ *are tangent to the plane* $\exp^\mu(NK_{(1,0,..,0)})$ *at* $(-1,0,0,..,0)$. *The horizontal collapsing,* $\exp^\mu(\gamma(s), 2N_\gamma(s)) = (-1,0,0,..,0)$ *is the only singularity, since* γ' *and* γ'' *being parallel to* E_{12} *implies that the singular set* $Sng(K,\mu) \subset E_{12}$ *by Proposition 8 of Section 6.*

Example 2. *The open arc of Example 1A can be extended to a simple closed curve with an appropriate* μ *to obtain examples with* $TIR < UR$. *Let* C_1 *be the unit circle centered at the origin. Given a small* $\varepsilon > 0$, *let* $q_1^+ = (\cos\varepsilon, \sin\varepsilon) \in C_1$ *and* $q_1^- = (\cos\varepsilon, -\sin\varepsilon)$. *Let* L^+ *and* L^- *be the tangent lines to* C_1 *at* q_1^+ *and* q_1^-, *respectively. Given a large* ℓ, *take* $q_2^+ \in L^+$ *so that the line segment between* q_1^+ *and* q_2^+ *has length* ℓ *and the* $y-$*coordinate* q_2^+ *is larger than of* q_1^+. *Take* $q_2^- \in L^-$ *in a symmetric manner with respect to the* $x-$*axis. Let* C_2 *be the circle tangent to* L^+ *at* q_2^+ *and to* L^- *at* q_2^-. *Consider the continuously differentiable closed convex curve* $\overline{\gamma}$ *which is a concatenation of* C_1 *between* q_1^- *and* q_1^+, L^+ *between* q_1^+ *and* q_2^+, C_2 *between* q_2^+ *and* q_2^-, *and* L^- *between* q_2^- *and* q_1^-. *Let* γ *be the smooth closed curve which is the same as* $\overline{\gamma}$ *outside small* $(0 < \delta \ll \varepsilon)$ $\delta-$*neighborhoods* $U_i^\pm$ *of* $q_i^\pm$, *such that the curvature is strictly monotone on each* $U_i^\pm$, *and* γ *is symmetric with respect to the* $x-$*axis. Parametrize* $\gamma(s)$ *with the domain* $[-A, A]$, $\gamma(0) = (1,0)$, *arclength* s, *and take* $K = \gamma([-A,A])$.

We will construct μ *so that* $\mu(-s) = \mu(s)$. *Let* $\mu = \cos\frac{s}{2}$ *for* $|s| \leq 2\varepsilon$. *For small* $\varepsilon > 0$, $\mu(2\varepsilon) \approx 1 - \frac{\varepsilon^2}{2}$, $\mu'(2\varepsilon) \approx -\frac{\varepsilon}{2}$, *and* $\mu''(2\varepsilon) \approx -\frac{1}{4}\left(1 - \frac{\varepsilon^2}{2}\right)$. *By taking* ℓ *sufficiently large, one can extend* μ *smoothly to* $[0, A]$ *so that* $\frac{-1}{4} \leq \mu'' \leq \frac{1}{20}$, $-\varepsilon \leq \mu' \leq 0$, *and* $\frac{1}{4} \leq \mu \leq 1$ *over* $[2\varepsilon, \ell]$, *and* $\mu \equiv c_0 \geq \frac{1}{4}$ *on* $[\ell - 1, A]$. *Observe that* $\gamma(\ell)$ *is on* L^+ *before* q_2^+, *and* $|\mu'| \leq \varepsilon$ *on all of* $[-A, A]$.

On $[0, \varepsilon - \delta] : \Delta(\kappa, \mu) = 0$, $\Lambda(\kappa,\mu) = \frac{1}{4}$, $FocRad^0(\gamma(s), \mu(s)) = 2$, *and* $\frac{4}{\varepsilon} \leq |\mu'(s)|^{-1} = FocRad^-(\gamma(s), \mu(s))$. *Moreover, for all* $s \in [0, \varepsilon - \delta]$, $(-1, 0) = \exp^\mu(\gamma(s), 2(-\cos s, -\sin s))$. *Hence,* $\exp^\mu$ *is singular and not injective along the* $R = 2$ *curve in* NK *and* $TIR(K,\mu) \leq 2$.

On $(\varepsilon - \delta, \varepsilon + \delta) : \Delta(\kappa,\mu) = \mu\left(\mu'' + \frac{1}{4}\kappa^2\mu\right) < 0$, *since* κ *is decreasing from* 1 *to* 0, *and* $\mu = \cos\frac{s}{2}$. *Hence,* $FocRad^0(\gamma(s), \mu(s)) = FocRad^-(\gamma(s), \mu(s)) \geq \frac{1}{\varepsilon}$.

On $[\varepsilon + \delta, \ell]$, $\kappa \equiv 0$. *Hence,* $\Lambda(\kappa, \mu) = \frac{1}{2}(\mu^2)'' = \mu\mu'' + (\mu')^2 \leq \frac{1}{20} + \varepsilon^2 \leq \frac{1}{16}$, *to conclude that* $FocRad^0(\gamma(s), \mu(s)) = FocRad^-(\gamma(s), \mu(s)) \geq 4$. *Observe that when* $\mu\mu'' + (\mu')^2 < 0$, *both pointwise radii are equal to* $|\mu'(s)|^{-1}$.

On $[\ell - 1, A]$, $\mu \equiv c_0$. $\Delta(\kappa, \mu) = \frac{\kappa^2 c_0^2}{4}$, $\Lambda(\kappa,\mu) = \kappa^2 c_0^2$ *and* $FocRad^0(\gamma(s), \mu(s))$ $= FocRad^-(\gamma(s), \mu(s)) \geq \frac{R_2}{c_0}$ *where* R_2 *is the radius of* C_2.

Overall, $FocRad^0(K,\mu) = 2$ *controlled by* C_1 *part and* $FocRad^-(K,\mu) \geq 4$. *For the double critical points* p *and* q *on* γ, $\cos\alpha(p,q) = -R\mu'(p)$, *and* $|\mu'(p)| \leq \varepsilon$. *By taking* $\varepsilon > 0$ *sufficiently small and* ℓ *sufficiently large, one can keep* $\alpha(p,q)$ *close to* $\frac{\pi}{2}$ *and* $\frac{1}{2}DCSD \geq 5$. *By Proposition 5(ii):*

$DIR(K,\mu) = TIR(K,\mu) = 2 < 4 \leq UR(K,\mu)$.

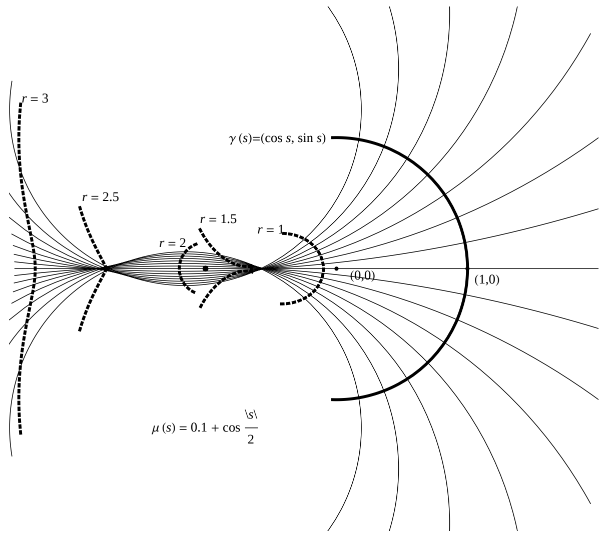

FIGURE 8. Compare the normal exponential maps from a portion of the unit circle with $\mu(s) = t + \cos\frac{s}{2}$ for $t = 0.1$ and $t = -0.1$ (Figure 9) with $t = 0$ (Figure 7). The diagrams also show the curves of type $\exp^{\mu}(\gamma(s), rN(s))$ for some choices of r. Figures 7-9 together show the instability of DIR under small perturbations.

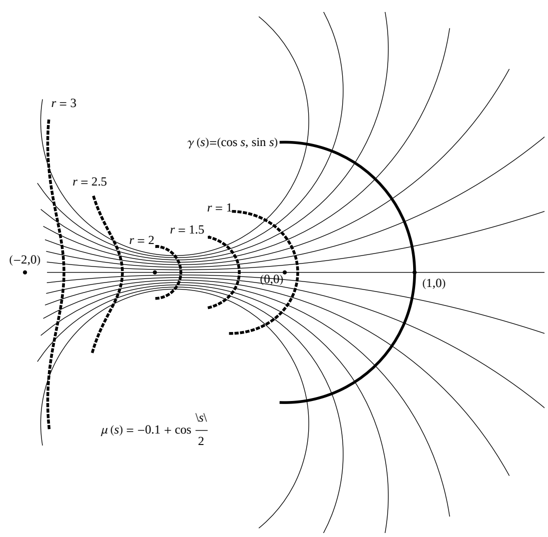

FIGURE 9.

EXAMPLE 3. *Figures 8 and 9. Let $\varepsilon, \ell, \gamma$ and μ be as in Example 2, and $\mu_t(s) = t + \mu(s) = t + \cos\frac{s}{2}$. For small $t > 0$, and $|s| < \varepsilon - \delta$, and $\kappa = 1$,*

$$\Delta(\kappa, \mu_t) = \mu_t\left(\mu_t'' + \frac{1}{4}\mu_t\right) > 0$$

$$\Lambda(\kappa, \mu_t) = \frac{1}{2}(\mu_t^2)'' + \frac{1}{2}\mu_t^2 + \mu_t\sqrt{\Delta(\kappa, \mu_t)} > \frac{1}{4}$$

$$FocRad^{-}(\gamma(s), \mu_t(s)) = FocRad^{0}(\gamma(s), \mu_t(s)) < 2$$

On the interval $(\varepsilon-\delta, \varepsilon+\delta)$, $\mu = \cos\frac{s}{2}$, *but* κ *starts to decrease to* 0 *and* Δ *becomes negative.* $\mu_t'' + \frac{1}{4}\kappa^2\mu_t = \mu'' + \frac{1}{4}\kappa^2(\mu+t) = \frac{1}{4}\left(\mu(\kappa^2-1)+t\kappa^2\right)$ *should have* 0 *as a regular value for almost all small* t *to secure that* $FocRad^- = FocRad^0$, *see the proof of Lemma 6. The effects of* t *on the remainder of* γ *and* $DCSD$ *are small. Hence, for almost all small* $t>0$, $DIR(K,\mu_t) = TIR(K,\mu_t) = UR(K,\mu_t) < 2$.

For small $t<0$ *and* $|s|<2\varepsilon$:

$$\Delta(\kappa,\mu_t) = \mu_t\left(\mu_t'' + \frac{1}{4}\kappa^2\mu_t\right) < 0$$

$$FocRad^0(\gamma(s),\mu_t(s)) = FocRad^-(\gamma(s),\mu_t(s)) \geq \frac{1}{\varepsilon}$$

The effects of t *on the remainder of* γ *and* $DCSD$ *are small. For all small* $t<0$:

$$FocRad^0(K,\mu_t) = FocRad^-(K,\mu_t) \geq 3$$

$$DIR(K,\mu_t) = TIR(K,\mu_t) = UR(K,\mu_t) \geq 3$$

We see that TIR *and* DIR *are not upper semicontinuous:*

$$\liminf_{t\to 0^-} DIR(K,\mu_t) = \liminf_{t\to 0^-} TIR(K,\mu_t) \geq 3 > 2 = TIR(K,\mu) = DIR(K,\mu)$$

$$\lim_{n\to\infty} UR(K,\mu_{t_n}) \leq 2 < 4 \leq UR(K,\mu) \text{ for some sequence } 0<t_n\to 0.$$

Example 4. *Figure 10. Let* $\gamma(s) = (\cos s, \sin s) : \mathbf{R} \to K \subset \mathbf{S}^1 \subset \mathbf{R}^2$ *and* $\mu(s) = 1-\frac{s^2}{8}$ *for* $|s|<1$. *Observe that* $0 < \left(\cos\frac{s}{2}\right) - \left(1-\frac{s^2}{8}\right) = o(s^3)$ *for* $s\neq 0$.

$$\forall s,\ \Delta(\kappa,\mu) = \mu\left(\mu'' + \frac{1}{4}\mu\right) = \frac{1}{256}s^2(s^2-8) \leq 0$$

$$\forall s,\ \Lambda(\kappa,\mu) = \begin{cases} \frac{1}{4} & \text{if } s=0 \\ \text{not a real number} & \text{if } s\neq 0 \end{cases}$$

$$\forall s,\ FocRad^0(\gamma(s),\mu(s)) = \begin{cases} 2 & \text{if } s=0 \\ \frac{4}{|s|} & \text{if } s\neq 0 \end{cases}$$

$$\forall s,\ FocRad^-(\gamma(s),\mu(s)) = \frac{1}{|\mu'(s)|} = \frac{4}{|s|}$$

$$FocRad^0(K,\mu) = 2 < 4 = FocRad^-(K,\mu)$$

Since $\mu'(0)=0$, $\exp^\mu(NK_{(1,0)})$ *is the* $x-axis$. *For* $s\neq 0$, $exp^\mu(NK_{\gamma(s)}\cap W)$ *is a circle of radius* $\left|\frac{\mu}{2\mu'}\right| = \frac{8-s^2}{4s}$ *and with center* $(\cos s,\sin s) + \frac{8-s^2}{4s}(-\sin s,\cos s)$. $exp^\mu(NK_{\gamma(s)}\cap W)$ *intersects* $\mathbf{S}^1$ *perpendicularly at both* $(\cos s,\sin s)\in K$ *and* $(\cos\theta(s),\sin\theta(s))\notin K$ *where* $\theta(s):(-1,1)\to(\frac{\pi}{2},\frac{3\pi}{2})$ *is a smooth function, and*

$$\theta(s) = s + 2\arctan\frac{8-s^2}{4s} \text{ and } \theta'(s) = \frac{s^2(s^2-8)}{s^4+64},\ \text{for } s>0.$$

This shows that $\theta(s)$ *is an injective function, but* $\theta'(0)=0$. *All of the circles* $exp^\mu(NK_{\gamma(s)}\cap W)$ *are disjoint from each other and the* $x-axis$. *As* $s\to 0$, *the pointwise focal radii tend to* ∞, *and the circles converge to the* $x-axis$. *Consequently, for all* ε *with* $0<\varepsilon<1$, $exp^\mu((\cos s,\sin s), R(-\cos s,-\sin s))$ *is injective and a homeomorphism onto its image for* $|s|<\varepsilon$ *and* $|R| < \frac{4}{\varepsilon} = \inf\frac{1}{|\mu'|}$. *However,* $\exp^\mu$ *is singular at one isolated point* $(q,Rv) = ((1,0),2(-1,0))$, $p = \exp^\mu((1,0),2(-1,0)) = (-1,0)$. *Hence, there exists a non-closed curve with:*

$2 = DIR(K,\mu) < TIR(K,\mu) = \frac{4}{\varepsilon}$ *and* $0<\varepsilon<1$.

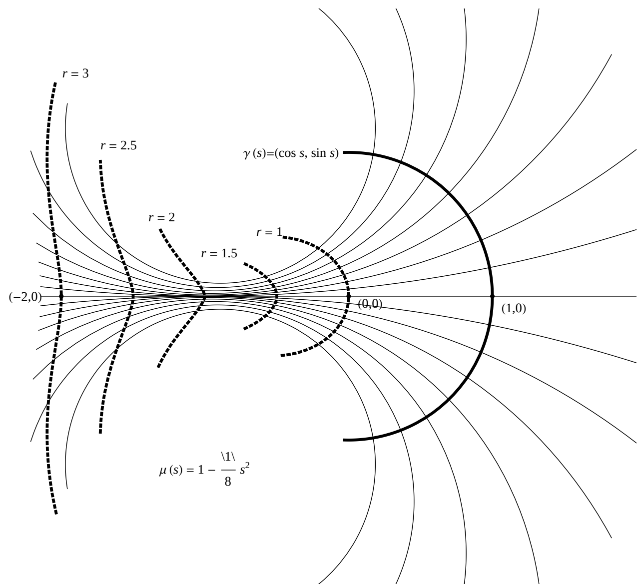

FIGURE 10. $\gamma(s) = (\cos s, \sin s)$ and $\mu(s) = 1 - \frac{s^2}{8}$. This figure shows an exponential map which is a local homeomorphism but not a local diffeomorphism near $(-1, 0)$. See Example 4.

EXAMPLE 5. *Construct γ and μ exactly in the same fashion as in Example 2, with $\mu(s) = 1 - \frac{s^2}{8}$ instead of $\cos \frac{s}{2}$ on $(-2\varepsilon, 2\varepsilon)$. On $[\delta - \varepsilon, \varepsilon - \delta]$ one has $\Delta(\kappa, \mu) = -\frac{1}{256} s^2(s^2 - 8) \leq 0$, $\Lambda(\kappa, \mu)(0) = \frac{1}{4}$. For $s = 0$, $FocRad^0(\gamma(0), \mu(s)) = 2$, and $FocRad^-(\gamma(0), \mu(s)) = \infty$. For $s \neq 0$, $FocRad^0(\gamma(s), \mu(s)) = FocRad^-(\gamma(s), \mu(s)) = \frac{1}{|\mu'(s)|} \geq \frac{2}{\varepsilon}$. The remaining estimates are the same as in Example 2. Overall, $FocRad^0(K, \mu) = 2$ controlled only by one point, $\gamma(0)$, and $FocRad^-(K, \mu) \geq 4$. Observe that there is only one point (q, Rv) where $p = \exp^\mu(q, Rv)$, $F_p''(s) = 0$, and $R < 3$, namely $((1, 0), 2(-1, 0))$. Suppose that $3 > TIR(K, \mu)$ and repeat the proof of Proposition 4. Since, $\frac{1}{2} DCSD \geq 5$, the only possibilities left are the Cases 1 and 5. If both $y_0 = z_0 = \gamma(0)$, then this would contradict the $\exp^\mu$ being a local homeomorphism as discussed in Example 4. If $z_0 \neq \gamma(0)$, then one still can repeat the argument of Case 5, by finding $\mu-$closest point q_1 to p_1 by using the fact that $\exp^\mu$ is a local homeomorphism again, to obtain a double critical point, which is not the case. This shows that $DIR(K, \mu) = 2 < 3 \leq TIR(K, \mu)$.*

EXAMPLE 6. *Figures 11 and 12. Let $\gamma(s) = (\cos s, \sin s) : \mathbf{R} \to K \subset \mathbf{S}^1 \subset \mathbf{R}^2$ and $\mu_t(s) = t + 1 - \frac{s^2}{8}$ for $|s| < 1 = \varepsilon$. For small $t > 0$,*

$$\Delta(\kappa, \mu_t) = \mu_t \left(\mu_t'' + \frac{1}{4} \mu_t \right) > 0 \textit{ for } |s| < \sqrt{8t}$$

$$\Lambda(\kappa, \mu_t) > \frac{1}{4} \textit{ for } |s| < \sqrt{8t}$$

$$\Delta(\kappa, \mu_t) < 0 \textit{ for } \sqrt{8t} < |s| < 1$$

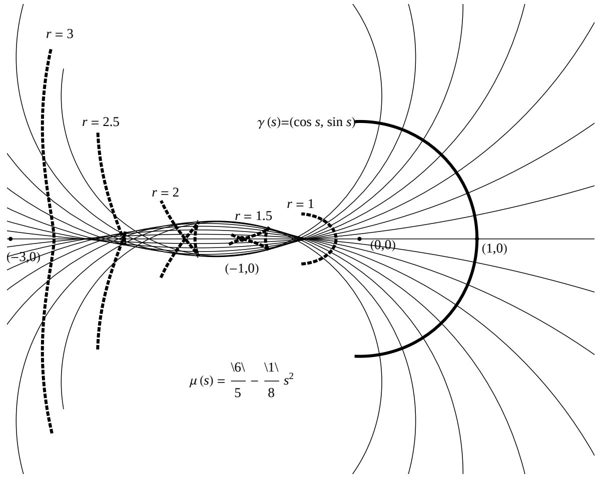

FIGURE 11. Compare the normal exponential maps from a portion of the unit circle with $\mu(s) = t + 1 - \frac{1}{8}s^2$ for $t = 0.2$ and $t = -0.05$ (Figure 12) with $t = 0$ (Figure 10). The diagrams also show the curves of type $\exp^{\mu}(\gamma(s), rN(s)$ for some choices of r. Figures 10-12 together show the instability of TIR under small perturbations.

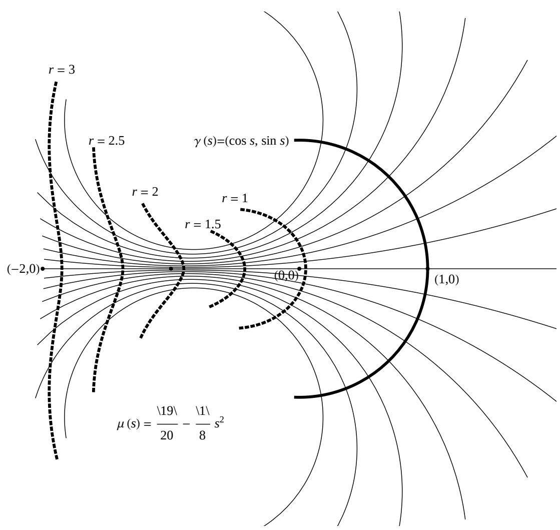

FIGURE 12. The normal exponential map from a portion of the unit circle with $\mu(s) = 0.95 - \frac{1}{8}s^2$ is a local diffeomorphism.

$$FocRad^{-}(\gamma(s), \mu_t(s)) = FocRad^{0}(\gamma(s), \mu_t(s)) < 2 \ for \ |s| < \sqrt{8t}$$

$$DIR(K, \mu_t) = TIR(K, \mu_t) < 2$$

For small $t<0$ and $|s|<1$:

$$\Delta(\kappa,\mu_t)=\mu_t\left(\mu_t''+\frac{1}{4}\mu_t\right)<0$$
$$FocRad^0(\gamma(s),\mu_t(s))=FocRad^-(\gamma(s),\mu_t(s))=\frac{4}{|s|}\geq 4$$

Suppose that there is a double critical pair (p,q) for (K,μ). Then, both $\alpha(p,q)$ and $\alpha(q,p)$ must be larger than or equal to $\frac{\pi}{2}$, by Lemma 1. On $\gamma(s)$, $\mu(s)$ is increasing as $|s|\to 0$. Hence, gradμ points in the direction of $\gamma(0)=(1,0)$, and grad$\mu(0)=0$. For any two points p and q on $\gamma(s)$, $|s|<1$, the line segment joining them can not make angle larger than or equal to $\frac{\pi}{2}$ with gradμ at both end points, at least one of them is acute. Hence, there is no double critical pair on γ. For $t<0$,

$$DIR(K,\mu_t)=TIR(K,\mu_t)=4.$$

Combining with Example 4, we see that TIR and DIR have different semicontinuity properties:

$$\lim_{t\to 0^-}DIR(K,\mu_t)=4>2=DIR(K,\mu)\geq \limsup_{t\to 0^+} DIR(K,\mu_t)$$
$$\lim_{t\to 0^-}TIR(K,\mu_t)=4=TIR(K,\mu)>2\geq \limsup_{t\to 0^+} TIR(K,\mu_t)$$

6. AIR and TIR

The almost injectivity radius $AIR(K,\mu,\mathbf{R}^n)$ is

$$\sup\left\{\begin{array}{c} r:\exp^\mu:U(r)\to U_0(r) \text{ is a homeomorphism where } U(r) \text{ is an open} \\ \text{and dense subset of } D(r), \text{ and } U_0(r) \text{ is an open subset of } \mathbf{R}^n.\end{array}\right\}.$$

We observe that $\exp^\mu : D(r)\to O(K,\mu r)$ is a smooth onto map, where both $D(r)$ and $O(K,\mu r)$ are open subsets (for $r>0$) of $n-$dimensional manifolds. For $0<r<AIR(K,\mu)$ and all nonempty open subsets V of $D(r)$, $\exp^\mu(V\cap U(r))$ is a nonempty open subset of $O(K,\mu r)$, and $\exp^\mu(V\cap U(r))$ is dense in $\exp^\mu(V)$. $\exp^\mu(V)$ is not necessarily open in $O(K,\mu r)$ when V contains singular points of $\exp^\mu$, see Figure 7 around $(-1,0)$.

Proposition 6. *If $p_0=\exp^\mu(q_1,R_1v_1)=\exp^\mu(q_2,R_2v_2)$ with $v_i\in UNK_{q_i}$ for $i=1,2$, and $0\leq\sqrt{G(p_0)}=R_2<R_1$, then $AIR(K,\mu)<R_1$.*

Proof. Let $R_0=AIR(K,\mu)$. For $q\in K$ and $r>0$, let $A(q,r)$ denote the connected component of $B(q,r;\mathbf{R}^n)\cap K$ containing q and $A^c(q,r)=K-\overline{A(q,r)}$. $A(q,r)$ is an open arc for small r. First, we will show that $R_1\geq R_0$.

Suppose that $R_1<R_0$. Let $\varepsilon=\frac{1}{3}\min(R_0-R_1,R_1-R_2)>0$. Choose $\sigma>0$ such that

$$0<\sigma<\mu(q_1)\varepsilon \text{ and}$$
$$\max\left\{\mu(q):q\in\overline{A(q_1,\sigma)}\right\}\leq\left(1+\frac{\varepsilon}{R_1}\right)\min\left\{\mu(q):q\in\overline{A(q_1,\sigma)}\right\}.$$

We assert that $q_2 \in A^c(q_1, \sigma)$, since the assumption of $q_2 \in \overline{A(q_1, \sigma)}$ leads to a contradiction as follows:

$$\begin{aligned}
\sigma &\geq \|q_1 - q_2\| \\
&\geq \|q_1 - p_0\| - \|q_2 - p_0\| \\
&\geq R_1\mu(q_1) - R_2\mu(q_2) \\
&\geq R_1\mu(q_1) - R_2\left(1 + \frac{\varepsilon}{R_1}\right)\mu(q_1) \\
&\geq \mu(q_1)\left(R_1 - R_2 - \frac{\varepsilon R_2}{R_1}\right) \\
&\geq \mu(q_1)\left(3 - \frac{R_2}{R_1}\right)\varepsilon \\
&> 2\mu(q_1)\varepsilon.
\end{aligned}$$

We are given that $G(p_0) = \min_{q\in K} F_{p_0}(q)$, and

$$\sqrt{G(p_0)} = R_2 < R_1 = \frac{\|p_0 - q_1\|}{\mu(q_1)} = \sqrt{F_{p_0}(q_1)}.$$

There exists a small open neighborhood V_0 of p_0 in $\mathbf{R}^n$, such that $\overline{V_0}$ is compact with

$$\begin{aligned}
&\overline{V_0} \subset B(q_1, (R_1 + \varepsilon)\mu(q_1); \mathbf{R}^n) \cap B(q_2, (R_2 + \varepsilon)\mu(q_2); \mathbf{R}^n) \text{ and} \\
&\forall p \in \overline{V_0},\ \sqrt{G(p)} \leq R_2 + \varepsilon < R_1 - \varepsilon \leq \frac{\|p - q_1\|}{\mu(q_1)} = \sqrt{F_p(q_1)}.
\end{aligned}$$

Therefore, there exists $0 < \sigma_0 < \sigma$ such that for every $p \in \overline{V_0}$, each $\mu-$closest point $q_2(p)$ of K to p satisfies that $q_2(p) \in A^c(q_1, \sigma_0)$, by an argument similar to above for q_2 with $\varepsilon/3$ replacing ε in the choice of σ_0. We choose r such that $R_1 + 2\varepsilon < r < R_0$ and take:

$$\begin{aligned}
D_1 &= \{(q, w) \in NK : q \in A(q_1, \sigma_0) \text{ and } \|w\| < r\}, \\
D_2 &= \{(q, w) \in NK : q \in A^c(q_1, \sigma_0) \text{ and } \|w\| < r\}, \text{ and} \\
V_i &= \left(\exp^\mu \mid D_i\right)^{-1}(V_0) \text{ for } i = 1, 2.
\end{aligned}$$

Both V_1 and V_2 are open in NK, $V_1 \cap V_2 \subset D_1 \cap D_2 = \varnothing$, but $(q_i, R_i v_i) \in V_i \neq \varnothing$ for $i = 1, 2$. The way σ_0 and r were chosen above implies that $\overline{V_0} \subset \exp^\mu(D_2)$ and $\exp^\mu(V_2) = V_0$. Consequently, $\exp^\mu(V_2 \cap U(r))$ is a nonempty, open and dense subset of V_0. However, $\exp^\mu(V_1 \cap U(r))$ is a nonempty, open (but not necessarily dense) subset of V_0. Hence,

$$\begin{aligned}
\exp^\mu(V_1 \cap U(r)) \cap \exp^\mu(V_2 \cap U(r)) &\neq \varnothing, \\
\text{but } V_1 \cap V_2 &= \varnothing.
\end{aligned}$$

This contradicts the definition of AIR. Hence, $AIR(K, \mu) = R_0 \leq R_1$.

For sufficiently small $\delta > 0$, there is δ' such that $\exp^\mu(q_1, (R_1 - \delta)v_1) = p_1$ satisfies that $\sqrt{G(p_1)} = R_2 + \delta' < R_1 - \delta$. There exists $q_3 \in K$ and $v_3 \in UNK_{q_3}$ such that $p_1 = \exp^\mu(q_3, (R_2 + \delta')v_3)$. By the preceding part of the proof, $AIR(K, \mu) \leq R_1 - \delta < R_1$. □

Corollary 2. *i. If* $R < AIR(K,\mu)$, *then* $\exp^\mu(\partial D(R)) = \partial O(K,\mu R)$.

ii. If $\exp^\mu(q_1, R_1 v_1) = \exp^\mu(q_2, R_2 v_2)$ *and* $R_i < AIR(K,\mu)$ *for* $i = 1$ *and* 2, *then* $R_1 = R_2$.

iii. If $R_1 < R_2 < AIR(K,\mu)$, *then* $\exp^\mu(\partial D(R_1)) \cap \exp^\mu(\partial D(R_2)) = \varnothing$.

Proof. $\exp^\mu(D(R)) = O(K,\mu R) = G^{-1}([0,R^2))$ and all are open subsets of $\mathbf{R}^n$, for all $R > 0$, by Corollary 1 of Proposition 1.

i. If $p \in \partial O(K,\mu R)$ then $G(p) = R^2$. Hence, $\partial O(K,\mu R) \subset \exp^\mu(\partial D(R))$. If there is $p \in \exp^\mu(\partial D(R))$ which is an interior point of $O(K,\mu R)$, then by Proposition 6, one would have $R > AIR(K,\mu)$.

ii and iii immediately follow Proposition 6, and the fact that for every p in $O(K,\mu R)$, there exists $q \in K$ and $v \in UNK_q$ such that $p = \exp^\mu(q, rv)$ for some $r = \sqrt{G(p)} < R$. □

Proposition 7. *i.* $AIR(K,\mu) < \left(\max_{q\in K} \|grad\mu(q)\|\right)^{-1} < \infty$, *if* μ *is not constant.*

ii. $AIR(K,\mu) \le \left(c_0 \cdot \max_{q\in K} \kappa(q)\right)^{-1} < \infty$, *if* $\mu = c_0$ *is constant.*

iii. $TIR(K,\mu) \le AIR(K,\mu) \le UR(K,\mu)$.

Proof. i. By Proposition 1(vi), $exp^\mu(NK_q \cap W) \cap K$ has a least two distinct points, if $grad\mu(q) \neq 0$. Let q' $(\neq q)$ be another point of this set. Then, $q' = \exp^\mu(q, Rv_1) = \exp^\mu(q', 0)$ for some $R \le \|grad\mu(q)\|^{-1}$. By Proposition 6, $AIR(K,\mu) < R$. Since K is compact, $\max_{q\in K} \|grad\mu(q)\|$ is attained on K.

ii. This is a part of the proof of (iii).

iii. First inequality follows the definitions.

Suppose there exists R such that $FocRad^-(K,\mu) < R < AIR(K,\mu)$. Then, there exists $p_1 = \exp^\mu(q_1, Rv_1)$, for some $v_1 \in UNK_{q_1}$ and $q_1 \in CP(p_1, -)$. As in the Claim 1 in the proof Proposition 4, $G(p_1) < R^2$, and $p_1 = \exp^\mu(q_2, R_2 v_2)$ for some $(q_2, R_2 v_2) \neq (q_1, Rv_1)$ with $R_2 < R$. This contradicts Corollary 2(ii). Consequently, $AIR(K,\mu) \le FocRad^-(K,\mu)$.

We prove (ii) at this stage. If $\mu = c_0$, a positive constant, then $\Delta(\kappa, c_0) = \frac{1}{4}\kappa^2 c_0^2 \ge 0$, $\Lambda(\kappa, c_0) = \kappa^2 c_0^2$. Since K is compact, there exists a point q_0 of K with maximal $\kappa(q_0) > 0$. $AIR(K,\mu) \le FocRad^-(K,\mu) \le (\kappa(q_0)c_0)^{-1} < \infty$. If μ is not constant, then $AIR(K,\mu) < \infty$ by (i).

Suppose that $\frac{1}{2}DCSD(K,\mu) = R_0 < AIR(K,\mu)$. Let $AIR(K,\mu) - R_0 = \varepsilon > 0$. Since K is compact, the set of critical points of Σ is a compact subset of $K \times K$. Let (q_3, q_4) be a minimal double critical pair for (K,μ), with p on the line segment $\overline{q_3 q_4}$ joining q_3 and q_4 such that $\|p - q_i\| = R_0 \mu(q_i)$ and $p = \exp^\mu(q_i, R_0 v_i)$ for $i = 3, 4$. By Lemma 1 with $c = 0$, $\alpha(q_3, p) \in \left[\frac{\pi}{2}, \pi\right]$. First, we consider the case $\alpha(q_3, p) > \frac{\pi}{2}$ where $\operatorname{grad}\mu(q_3) \neq 0$. By part (i) and Proposition 1(ii), $\alpha(q_3, p) \neq \pi$. The circular arc $\beta(s) = \exp^\mu(q_3, sv_3)$ is contained in the 2-plane containing q_3, p and q_4 and parallel to v_3. $\measuredangle(\beta'(0), u(q_3, p)) = \measuredangle(\beta'(R_0), u(p, q_4)) = \alpha(q_3, p) - \frac{\pi}{2} < \frac{\pi}{2}$. Since $\|q_i - p\| = \mu(q_i) R_0$ for $i = 3, 4$, one has $\|q_4 - \beta(R_0 + s)\| \le (R_0 - \lambda s)\,\mu(q_4) < R_0 \mu(q_4)$ for some $\lambda > 0$ and small enough $\delta > s > 0$. In the case of $\alpha(q_3, p) = \frac{\pi}{2}$, the last statement still holds since $\beta(s)$ traces the line segment $\overline{q_3 q_4}$. In all cases, choose $p_0 = \beta(R_0 + s_0)$ such that $0 < s_0 < \min(\varepsilon, \delta)$.

$$F_{p_0}(q_3) = (R_0 + s_0)^2 > (R_0 - \lambda s_0)^2 \ge F_{p_0}(q_4) \ge G(p_0) = F_{p_0}(q_5)$$

for some $q_5 \in K$. By Proposition 6, $AIR(K,\mu) < R_0 + s_0 < R_0 + \varepsilon$ which contradicts the initial assumptions. Hence, $AIR(K,\mu) = R_0 \le \frac{1}{2}DCSD(K,\mu)$. □

Proposition 8. *Let K_i denote the components of K. Let $\gamma_i : domain(\gamma_i) \to K_i$ be an onto parametrization of the component K_i with unit speed and $\mu_i(s) = \mu(\gamma_i(s))$. Then, the singular set $Sng^{NK}(K,\mu)$ of $\exp^\mu$ within $D(UR(K,\mu)) \subset NK$ is a graph over a portion of K: $Sng^{NK}(K,\mu) = \bigcup_i Sng_i^{NK}(K,\mu)$ and*

$$Sng_i^{NK}(K,\mu) = \left\{ \begin{array}{c} (\gamma_i(s), R_i(s)N_{\gamma_i}(s)) \in NK_i \textit{ where} \\ s \in domain(\gamma_i),\ \kappa_i(s) > 0, \\ \left(\mu_i'' + \frac{1}{4}\kappa_i^2\mu_i\right)(s) = 0, \textit{ and} \\ 0 < R_i(s) = \left(\left((\mu_i')^2 - \mu_i\mu_i''\right)(s)\right)^{-\frac{1}{2}} < UR(K,\mu) \end{array} \right\}$$

where κ_i and N_{γ_i} are the curvature and the principal normal of γ_i, respectively. $D(UR(K,\mu)) - Sng^{NK}(K,\mu)$ is connected in each component of NK, when $n \geq 2$.

Proof. We will prove it for connected K, and omit "i", since this is a local result. $R < UR(K,\mu) \leq \frac{1}{|\mu'(s)|}, \forall s$.

$$Sng^{NK}(K,\mu) = \left\{ \begin{array}{c} (q, Rv) : v \in UNK_q,\ R < UR(K,\mu) \\ \text{and the differential } d(\exp^\mu)(q, Rv) \text{ is singular} \end{array} \right\} \subset int(W).$$

For $q = \gamma(t)$, $v \in UN_q$, $p = \exp^\mu(q, Rv)$ and $R < FocRad^-(K,\mu)$:

$$0 \leq \frac{d^2}{ds^2}\, F_p(\gamma(s))\big|_{s=t} = \frac{2}{\mu^2(t)}\left(1 - \kappa R\mu\sqrt{1-(\mu' R)^2}\cos\beta - \frac{R^2}{2}(\mu^2)''\right)(t) \tag{6.1}$$

by Proposition 2, where $\beta = \measuredangle(\gamma''(t), u(q,p)^N)$ when both vectors are non-zero, and $\beta = 0$ otherwise. By proposition 5(ii),

$\exp^\mu$ is singular at (q, Rv) if and only if $F_p''(t) = 0$, when the equality holds in (6.1). For fixed q and v, there is only one possibility, a repeated root as Lemma 3(vi), to have a zero of (6.1) and keeping (6.1) non-negative for all $0 < R < UR(K,\mu)$.

Case 1: $\kappa(t) = 0$. The quadratic in (6.1) can not have a repeated root when $(\mu^2)''(t) > 0$ and it has no roots when $(\mu^2)''(t) \leq 0$. Hence, it has no solution with $R < UR(K,\mu)$, and $Sng^{NK}(K,\mu)$ has no part over zero curvature points of γ.

Case 2. $\kappa(t) \neq 0$, with $N_\gamma(t)$ denoting the principal normal of γ. If the expression in (6.1) were zero for $q = \gamma(t)$, $R > 0$ and a unit vector $v \neq N_\gamma(t)$ (that is $\cos\beta < 1$), then it would be negative for the same q and R but $v_1 = N_\gamma(t)$ (with $\cos\beta_1 = 1$), which would imply that $R \geq UR(K,\mu)$. This proves that Sng^{NK} must be in the direction of the normal N_γ. In order have a singular point at $(\gamma(t), Rv)$ and to satisfy (6.1), one must have $v = N_\gamma(t)$ ($\cos\beta = 1$) and there must be repeated roots as in Lemma 3(vi), which occur only when $\Delta(\kappa,\mu) = 0$:

$$\begin{aligned} \Delta(\kappa,\mu) &= \frac{1}{2}(\mu^2)'' + \frac{1}{4}\kappa^2\mu^2 - (\mu')^2 = \mu\mu'' + \frac{1}{4}\kappa^2\mu^2 = 0 \\ \Lambda(\kappa,\mu) &= \frac{1}{2}(\mu^2)'' + \frac{1}{2}\kappa^2\mu^2 = (\mu')^2 - \mu\mu'' \\ \frac{1}{R^2} &= \Lambda(\kappa,\mu)(t) > 0 \text{ when } \kappa(t) > 0. \end{aligned}$$

It is straightforward to show that points satisfying these conditions are the singular points of $\exp^\mu$ within $D(UR(K,\mu))$. If $\mu = c_0$ is constant and $\kappa > 0$, then $\Delta(\kappa,\mu) > 0$, and as R increases, the first zero of $F_p''(t)$ occurs at $R = c_0/\kappa(t)$ and becomes negative for $R > c_0/\kappa(t)$. Consequently, $Sng^{NK}(K,\mu) = \varnothing$ when μ is constant. Since K is compact, if μ is not constant then there are points where $\mu'' > 0$ and $\Delta > 0$. Hence, the domain of the graph Sng^{NK} is not all of K. Including the dimension $n = 2$, the complement $D(UR) - Sng^{NK}$ is connected in each component of NK. $\square$

PROPOSITION 9. $\exp^{\mu}$ *restricted to* $D(UR(K,\mu)) - Sng^{NK}(K,\mu)$ *is a diffeomorphism onto its image in* $\mathbf{R}^n$ *and* $AIR(K,\mu) = UR(K,\mu)$.

PROOF. Let $0 < R_1 < UR(K,\mu)$ be chosen arbitrarily. $\exp^{\mu}$ is a non-singular map (local diffeomorphism) on $D(R_1) - Sng^{NK}(K,\mu)$ which is an open subset of NK. Let $\mu_\varepsilon(s) = \mu(s) - \varepsilon$ for small $\varepsilon > 0$.

$\exists \varepsilon_0 > 0$ such that $\forall \varepsilon \in (0, \varepsilon_0)$, $\exp^{\mu_\varepsilon} : D(R_1) \to \mathbf{R}^n$ is a non-singular map by the following. $\Delta(\kappa, \mu_\varepsilon) = \mu_\varepsilon \left(\mu_\varepsilon'' + \frac{1}{4}\kappa^2 \mu_\varepsilon\right) = (\mu - \varepsilon)\left(\mu'' + \frac{1}{4}\kappa^2\mu - \frac{1}{4}\kappa^2\varepsilon\right)$. On the parts of K where $\mu'' + \frac{1}{4}\kappa^2\mu \leq 0$, and $\kappa > 0$, one has $\Delta(\kappa, \mu_\varepsilon) < 0$ and hence $\exp^{\mu_\varepsilon}$ is non-singular for all small $\varepsilon > 0$, by Propositions 3 and 5. On the parts of K where $\mu'' + \frac{1}{4}\kappa^2\mu \leq 0$ and $\kappa = 0$, $\exp^{\mu_\varepsilon}$ is non-singular within radius of $UR(K,\mu_\varepsilon) \leq FocRad^{-}(K,\mu_\varepsilon)$, see the Case 1 in the proof of Proposition 8. On the parts of K where $\mu'' + \frac{1}{4}\kappa^2\mu > 0$, one has $\Lambda(\kappa,\mu)^{-\frac{1}{2}} \geq UR(K,\mu)$. Observe that $\Delta(\kappa,\mu_\varepsilon)(s_0) > 0$ implies that $\Delta(\kappa,\mu)(s_0) > 0$, and by Proposition 3(ii) both inequalities must be valid at some common points on K. By continuity, $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0,\varepsilon_0)$, $\Lambda(\kappa,\mu_\varepsilon)^{-\frac{1}{2}} \geq R_1$ and $Sng^{NK}(\mu_\varepsilon) \cap D(R_1) = \varnothing$, by Propositions 3, 8, and Definitions 4, 9. Consequently, $\exp^{\mu_\varepsilon} : D(R_1) \to \mathbf{R}^n$ is a non-singular map.

Suppose that $\exp^{\mu}$ is not one-to-one on $D(R_1) - Sng^{NK}(K,\mu)$, and there exist $(q_i, w_i) \in D(R_1) - Sng^{NK}(K,\mu)$ for $i = 1,2$ such that $(q_1,w_1) \neq (q_2,w_2)$ but $\exp^{\mu}(q_1,w_1) = \exp^{\mu}(q_2,w_2)$. By the regularity of $\exp^{\mu}$ on $D(R_1) - Sng^{NK}(K,\mu)$, there exists open sets U_i such that $(q_i,w_i) \in U_i \subset D(R_1) - Sng^{NK}(K,\mu)$ for $i = 1,2$, $U_1 \cap U_2 = \varnothing$, $\exp^{\mu}(U_1) = \exp^{\mu}(U_2)$ and $\exp^{\mu} \mid U_i$ are diffeomorphisms. $\{\exp^{\mu_\varepsilon} : \varepsilon > 0\}$ converge uniformly to $\exp^{\mu}$ on $\overline{D(R_1)}$ as $\varepsilon \to 0^+$, by the definition of $\exp^{\mu}$. Since $\exp^{\mu_\varepsilon}(U_1)$ and $\exp^{\mu_\varepsilon}(U_2)$ are open subsets of $\mathbf{R}^n$ and $\exp^{\mu}(U_1) = \exp^{\mu}(U_2)$, $\exists \varepsilon_1 > 0, \forall \varepsilon \in (0,\varepsilon_1)$, $\exp^{\mu_\varepsilon}(U_1) \cap \exp^{\mu_\varepsilon}(U_2) \neq \varnothing$. Consequently, $\exp^{\mu_\varepsilon} : D(R_1) \to \mathbf{R}^n$ is not injective. By Proposition 5(iii), $DIR(K,\mu_\varepsilon) = \frac{1}{2}DCSD(K,\mu_\varepsilon) \leq R_1, \forall \varepsilon \in (0, \min(\varepsilon_0,\varepsilon_1))$. There exist pairs of points $(x_\varepsilon, y_\varepsilon) \in K \times K$ with $x_\varepsilon \neq y_\varepsilon$, $grad\Sigma_\varepsilon(x_\varepsilon,y_\varepsilon) = 0$, and $\frac{\|x_\varepsilon - y_\varepsilon\|}{\mu(x_\varepsilon)+\mu(y_\varepsilon)} = \frac{1}{2}DCSD(K,\mu_\varepsilon)$ where $\Sigma_\varepsilon : K \times K \to \mathbf{R}$ defined by $\Sigma_\varepsilon(x,y) = \|x-y\|^2 (\mu_\varepsilon(x) + \mu_\varepsilon(y))^{-2}$. By compactness and taking convergent subsequences (and using x_j, y_j and μ_j for simplifying the subindices), there exists $(x_j,y_j) \to (x_0,y_0) \in K \times K$ with $grad\Sigma(x_0,y_0) = 0$. Suppose that $x_0 = y_0$. As $R_j = \|x_j - y_j\| (\mu(x_j) + \mu(y_j))^{-1} \to 0$, one has $\cos\alpha(x_j,y_j) = -R_j \left|\mu_j'(x_j)\right| = -R_j |\mu'(x_j)| \to 0$, which means that the line through x_j and y_j is making an angle close to $\pi/2$ with K at x_j and y_j. On the other hand, $(x_j,y_j) \to (x_0,x_0)$ implies that the same lines are converging to a line tangent to K. Both can not happen simultaneously. Hence, $x_0 \neq y_0$, and (x_0,y_0) is a critical pair for (K,μ). By the definition of $DCSD$ and continuity, $\frac{1}{2}DCSD(K,\mu) \leq \frac{\|x_0-y_0\|}{\mu(x_0)+\mu(y_0)} \leq R_1$. However, this contradicts our initial assumption of $R_1 < UR(K,\mu) \leq \frac{1}{2}DCSD(K,\mu)$. Finally, $\forall R_1 < UR(K,\mu)$, $\exp^{\mu}$ is one-to-one on $D(R_1) - Sng^{NK}(K,\mu)$, and it is a non-singular map onto an open subset of $\mathbf{R}^n$. This proves that $\exp^{\mu} \mid D(UR(K,\mu)) - Sng^{NK}(K,\mu)$ is a diffeomorphism onto its image. $Sng^{NK}(K,\mu)$ has an empty interior, since it is a subset of a one-dimensional graph over a subset of K. By the definitions and Proposition 7, $AIR(K,\mu) = UR(K,\mu)$. $\square$

Corollary 3. *Let (K,μ) be given and $\mu_\varepsilon(s)=\mu(s)-\varepsilon$. For a given $0<R_1<UR(K,\mu)$, $\exists\varepsilon'>0$ such that $\forall\varepsilon\in(0,\varepsilon')$, $\exp^{\mu_\varepsilon}:D(R_1)\to O(K,\mu_\varepsilon R_1)$ is a diffeomorphism. The diffeomorphisms $\exp^{\mu_\varepsilon}$ converge uniformly to the (possibly singular) map $\exp^\mu$ as $\varepsilon\to 0^+$, on $\overline{D(R_1)}$.*

Proof. This follows the proof of Proposition 9. First, the regularity part is done in the same way. Then, one supposes that such ε' does not exist, and for all $j\in\mathbf{N}^+$, there exist $0<\varepsilon_j\le\frac{1}{j}$ with a non-singular and non-injective map $\exp^{\mu_{\varepsilon_j}}:D(R_1)\to\mathbf{R}^n$. One follows the proof above again, by using the limits of subsequences of double critical pairs of (K,μ_{ε_j}), to obtain a double critical pair for (K,μ) to contradict $R_1<UR(K,\mu)\le\frac{1}{2}DCSD(K,\mu)$. □

Proposition 10. *For a given (K,μ) and $q\in K$, let*

$$\begin{aligned} Sng &= \exp^\mu(Sng^{NK}),\\ A_q &= \exp^\mu\left(NK_q\cap D(UR)\right),\ \text{and}\\ A_q^* &= \exp^\mu\left(NK_q\cap D(UR)-Sng^{NK}\right). \end{aligned}$$

Then, i. $O(K,\mu UR)-Sng$ has a codimension 1 foliation by A_q^*, which are (possibly punctured) spherical caps or discs.

ii. $\exp^\mu(D(UR)-Sng^{NK})=O(K,\mu UR)-Sng$.

iii. If $A_{q_1}\cap A_{q_2}\neq\varnothing$ for $q_1\neq q_2$ then q_1 and q_2 must belong to the same component of K, and A_{q_1} intersects A_{q_2} tangentially at exactly one point $p_0=\exp^\mu(q_1,r_1v_1)=\exp^\mu(q_2,r_2v_2)$ where $(q_i,r_iv_i)\in Sng^{NK}$, for $i=1,2$.

*iv. **Horizontal Collapsing Property**:*

Assume that $\exp^\mu(q_1,r_1v_1)=\exp^\mu(q_2,r_2v_2)=p_0$ for $r_1,\ r_2<UR(K,\mu)$, $v_i\in UNK_{q_i}$ with $(q_1,r_1v_1)\neq(q_2,r_2v_2)$. Then, q_1 and q_2 belong to the same component of K, which is denoted by K_1. Let $\gamma(s):\mathbf{R}\to K_1\subset\mathbf{R}^n$ be a unit speed parametrization of K_1 such that $\gamma(s+L)=\gamma(s)$ where L is the length of K_1, $N_\gamma(s)$ denotes the principal normal of γ, and $q_i=\gamma(s_i)$ for $i=1,2$ with $0\le s_1<s_2<L$. Then, $r_1=r_2$, $v_i=N_\gamma(s_i)$ for $i=1,2$, and $\exp^\mu(\gamma(s),r_1N_\gamma(s))=p_0$, $\forall s\in I$ where $I=[s_1,s_2]$ or $[s_2-L,s_1]$.

Proof. The logical order of the proof is different from the presentation order of the results.

For different components K_1 and K_2 of K, the open sets $O(K_1,\mu R)$ and $O(K_2,\mu R)$ are disjoint for $R<UR(K,\mu)$, otherwise one can obtain a contradiction with Propositions 8 and 9. $\exp^\mu\mid D(UR)-Sng^{NK}$ is a diffeomorphism onto its image. $\exp^\mu\mid NK_q\cap D(UR)$ is also a diffeomorphism where the image A_q is an open (metric) disc of an $n-1$ dimensional plane or sphere. By Proposition 8, $\exp^\mu\left(Sng^{NK}\cap NK_q\right)$ contains at most one point denoted by q^*, if it exists. If such q^* does not exist, we use $\{q^*\}=\varnothing$. Let $A_q^*=A_q-\{q^*\}$. The diffeomorphism $\exp^\mu\mid D(UR)-Sng^{NK}$ carries the codimension 1 foliation of $D(UR)-Sng^{NK}$ by NK_q-Sng^{NK} to a codimension 1 foliation of $\exp^\mu(D(UR)-Sng^{NK})$ by A_q^*.

As in Corollary 3, let $\mu_\varepsilon(s)=\mu(s)-\varepsilon$ for small $\varepsilon>0$ and choose large $R_1<UR(K,\mu)$. By Proposition 9, $A_{q_1}^*\cap A_{q_2}^*=\varnothing$ for $q_1\neq q_2$. Therefore, $A_{q_1}\cap A_{q_2}\subset\{q_1^*,q_2^*\}$ for $q_1\neq q_2$. Suppose that A_{q_1} and A_{q_2} intersect transversally. For $n\ge 3$, $A_{q_1}\cap A_{q_2}$ would have infinitely many points, which is not the case. In all dimensions including $n=2$, take $R_1<UR(K,\mu)$ sufficiently large with $\{q_1^*,q_2^*\}\subset O(K,\mu R_1)$. By Corollary 3, $A_{q_1}(\mu_\varepsilon)\cap A_{q_2}(\mu_\varepsilon)=\varnothing$, for sufficiently small $\varepsilon>0$. In the limit as

$\varepsilon \to 0^+$, A_{q_1} and A_{q_2} can not intersect transversally, since transversality is an open condition. Hence, A_{q_1} and A_{q_2} are tangential to each other at q_1^* or q_2^* and there is only one point of intersection for $q_1 \neq q_2$, if the intersection is not empty. If both A_{q_1} and A_{q_2} are subsets of hyperplanes, then $A_{q_1} \cap A_{q_2} = \varnothing$ for $q_1 \neq q_2$.

From this point on, assume that $p_0 = \exp^\mu(q_1, r_1 v_1) = \exp^\mu(q_2, r_2 v_2)$, for $q_1 \neq q_2$. A_{q_1} and A_{q_2} must intersect tangentially at $p_0 \in \{q_1^*, q_2^*\}$, and q_1 and q_2 must belong to the same component of K, denoted by K_1. At least one of A_{q_i} is spherical. Choose A_{q_1} to be the subset of the sphere with center c_1 and the smaller radius σ_1 so that $\operatorname{grad}\mu(q_1) \neq 0$. Then, $\forall p \in A_{q_2}$, $\|c_1 - p\| \geq \sigma_1$. Let $\gamma(s) : \mathbf{R} \to K_1 \subset \mathbf{R}^n$ be a unit speed parametrization such that $\gamma(s+L) = \gamma(s)$ where L is the length of K_1, and $q_i = \gamma(s_i)$ for $i = 1, 2$ with $0 \leq s_1 < s_2 < L$. Let $\eta(s) = \exp^\mu(\gamma(s), Rv(s))$ be as in Lemma 5:

$$\eta'(s_1) \cdot (\eta(s_1) - c(s_1)) = \frac{\mu^3(s_1)}{4\mu'(s_1)} \frac{d^2}{ds^2}\, F_{\eta(s_1)}(\gamma(s))\Big|_{s=s_1} \text{ since } \mu'(s_1) \neq 0$$

$$\text{where } c(s_1) = c_1 = \gamma(s_1) - \frac{\mu(s_1)}{2\mu'(s_1)} \gamma'(s_1)$$

We will assume that $\mu'(s_1) > 0$, and work on the interval $[s_1, s_2]$. Otherwise, if $\mu'(s_1) < 0$, then one reparametrizes K_1 to traverse $\gamma([s_2 - L, s_1])$ with opposite orientation starting at q_1. Choose $R_1 < UR(K, \mu)$ sufficiently large with $\{q_1^*, q_2^*\} \subset O(K_1, \mu R_1)$.

Claim 1. There exists $\delta > 0$ such that
$\forall s \in (s_1, s_1 + \delta)$, $\forall p \in A_{\gamma(s)} \cap O(K_1, R_1 \mu)$, $d(c_1, p) \geq \sigma_1$.

For a given curve $(\gamma(s), Rv(s))$ in NK_1 as in Lemma 5, define

$$\begin{aligned}
&\eta_{Rv}(s) = \exp^\mu(\gamma(s), Rv(s)) \text{ and} \\
&f_{Rv}(s) = \|\eta_{Rv}(s) - c_1\|^2 \text{ so that} \\
&f_{Rv}(s_1) = \sigma_1^2 > 0 \text{ and } f'_{Rv}(s_1) = 2\eta'_{Rv}(s_1) \cdot (\eta_{Rv}(s_1) - c_1).
\end{aligned}$$

$$\begin{aligned}
&f'_{Rv}(s_1) > 0 \text{ if } \eta_{Rv}(s_1) \in A^*_{q_1} \\
&f'_{Rv}(s_1) = 0 \text{ if } \eta_{Rv}(s_1) = q_1^*
\end{aligned}$$

(In the next two statements, the compactness of $(A_{q_1} - B(q_1^*, \delta_1)) \cap \overline{O(K_1, R_1\mu)}$ is essential.)

$$\begin{aligned}
&\forall \delta_1 > 0, \exists \delta_2 > 0 \text{ such that} \\
&\text{if } \eta_{Rv}(s_1) \in (A_{q_1} - B(q_1^*, \delta_1)) \cap \overline{O(K_1, R_1\mu)} \text{ then } f'_{Rv}(s_1) \geq \delta_2 > 0.
\end{aligned}$$

$$\begin{aligned}
&\exists \delta > 0 \text{ such that } \delta \ll \min(R_1, r_1, R_1 - r_1) \text{ and} \\
&\text{if } \eta_{Rv}(s_1) \in (A_{q_1} - B(q_1^*, \delta_1)) \cap \overline{O(K_1, R_1\mu)} \text{ and } s \in (s_1, s_1 + \delta), \\
&\text{then } f_{Rv}(s) > \sigma_1^2.
\end{aligned}$$

Suppose there exists $Rv(s)$ with $\eta_{Rv}(s_1) \in A_{q_1} \cap B(q_1^*, \delta_1) \cap O(K_1, R_1\mu)$, $s' \in (s_1, s_1 + \delta)$ and $f_{Rv}(s') < \sigma_1^2$. Then, $A_{\gamma(s')}$ must intersect A_{q_1} near q_1^*. This intersection must be tangential as discussed above with q_1 and q_2. However, this cannot be the case when $f_{Rv}(s)$ takes values on both sides of σ_1^2. This proves the Claim 1:

$$\begin{aligned}&\exists\delta>0 \text{ such that}\\ \text{if } \eta_{Rv}(s_1)\in A_{q_1}\cap O(K_1,R_1\mu) &\text{ and } s\in(s_1,s_1+\delta) \text{ then } f_{Rv}(s)\geq\sigma_1^2, \text{ hence,}\\ &\forall s\in(s_1,s_1+\delta),\forall p\in A_{\gamma(s)}\cap O(K_1,R_1\mu),\|c_1-p\|\geq\sigma_1.\end{aligned}$$

Recall that $\forall p\in A_{q_2}$, $\|c_1-p\|\geq\sigma_1$ and A_{q_2} is tangent to A_{q_1} at p_0. To avoid any transversal intersections with A_{q_2}, $A_{\gamma(s)}$ must stay between the codimension 1 submanifolds (sphere or plane) containing A_{q_1} and A_{q_2}, respectively. This forces $A_{\gamma(s)}$ to be tangent to A_{q_1} at p_0 for $\forall s\in(s_1,s_1+\delta)$, which is still true on $[s_1,s_1+\delta]$ by taking closure.

Claim 2. $A_{\gamma(s)}$ is tangent to A_{q_1} at p_0 for $\forall s\in[s_1,s_2]$.

If $\mu'>0$ on $[s_1,s_2)$, then Claim 2 can be proved by a standard topology argument. It is also possible to have the existence of $s_3\in(s_1,s_2)$ with $\mu'>0$ on $[s_1,s_3)$ and $\mu'(s_3)=0$. Then, Claim 2 holds on $[s_1,s_3]$ by the same argument. Let $q_3=\gamma(s_3)$. A_{q_3} is a subset of a hyperplane $H=\{x\in\mathbf{R}^n: x\cdot\gamma'(s_3)=a_0\}$ dividing $\mathbf{R}^n$ into two half spaces and $A_{\gamma(s)}$ are tangent to A_{q_3} at p_0 for $\forall s\in[s_1,s_3)$. The spheres containing $A_{\gamma(s)}$ $(s\in[s_1,s_3))$ are on the same side of H as A_{q_1}, their centers are on the line ℓ perpendicular to H at p_0, and the set of their radii is $[\sigma_1,\infty)$. $\mu'(s_2)\neq 0$ and A_{q_2} is a subset of a sphere, since A_{q_2} and A_{q_3} are tangent at p_0. A_{q_1} and A_{q_2} must be on the opposite sides of H since the center of A_{q_2} is also on ℓ, and the radius of A_{q_2} is not less than the radius of A_{q_1}. By studying the function $g_{Rv}(s)=\gamma'(s_3)\cdot\exp(\gamma(s),Rv(s))$, and using the first characterization of F''_p in Lemma 5, in a similar proof to Claim 1, one can obtain that

$$\exists\delta'>0,\forall s\in(s_3,s_3+\delta'),\forall p\in A_{\gamma(s)}\cap O(K_1,R_1\mu),\ p\cdot\gamma'(s_3)\geq a_0.$$

To avoid any transversal intersections with A_{q_2}, $A_{\gamma(s)}$ must stay between the codimension 1 submanifolds (a sphere and a plane) containing A_{q_2} and A_{q_3}, respectively. This forces $A_{\gamma(s)}$ to be tangent to A_{q_3} as well as A_{q_1} at p_0 for $\forall s\in(s_3,s_3+\delta')$, which is still true on $[s_1,s_3+\delta']$ by taking closure and combining with above. $\mu'<0$ on $(s_3,s_3+\delta']$, since (i) any zero of μ' will give a hyperplane tangent to A_{q_3} which cannot happen, and (ii) any positive value of μ' will give a sphere whose center is on ℓ but on the same side of H as A_{q_1}, which cannot happen by continuity and $A_{\gamma(s)}\cap A_{\gamma(s')}=\{p_0\}$ for $s<s_3<s'$. One repeats the proof of Claim 1 by showing that f_{Rv} is decreasing with $\mu'<0$, and Lemma 5, to extend Claim 2 to $[s_1,s_2]$.

$p_0=\exp^\mu(\gamma(s),r(s)v(s))$ for some curve $(\gamma(s),r(s)v(s)):[s_1,s_2]\to NK_1$. Hence, $r(s)=\|\gamma(s)-p_0\|/\mu(s)\equiv r_1>0$ by the Corollary 2(ii), $v(s)=N_\gamma(s)$ and $(\mu')^2-\mu\mu''=r_1^{-2}$ on $[s_1,s_2]$ by Proposition 8. $\forall s\in[s_1,s_2]$, $q^*_{\gamma(s)}=p_0$, since $q^*_{\gamma(s)}$ is unique. One can extend $[s_1,s_2]$ to a maximal closed interval by requiring $p_0\in A_{\gamma(s)}$.

To summarize, if $\exp^\mu(q_1,r_1v_1)=\exp^\mu(q_2,r_2v_2)=p_0$, for r_1, $r_2<UR(K,\mu)$ and $v_i\in UNK_i$ for $i=1,2$, then (i) $r_1=r_2$, (ii) $\exp^\mu(\gamma(s),r_1N_\gamma(s))=p_0$, $\forall s\in[s_1,s_2]$, and (iii) $v_i=N_\gamma(s_i)$ for $i=1,2$. However, it is essential to observe that this can be done on one arc of γ between q_1 and q_2, not both, since we chose the interval $[s_1,s_2]$ in a particular way above.

Observe that $q^*_{\gamma(s)} = p_0$, $\forall s \in [s_1, s_2]$ or $[s_2 - L, s_1]$, if $p_0 \in A_{\gamma(s_1)} \cap A_{\gamma(s_2)}$. This proves that

$$\exp^\mu(Sng_i^{NK}) \cap \exp^\mu(NK_i \cap D(UR) - Sng_i^{NK}) = \varnothing \text{ and}$$
$$\exp^\mu(D(UR) - Sng^{NK}) = O(K, \mu UR) - Sng.$$

□

REMARK 4. *In the proof of Claim 1 above, it is essential that the fibers A_q are subsets of spheres and planes. $f_x(t) = x^2 t - t^3$, satisfies that $f'_x(0) = x^2 > 0$ except $x = 0$, but "$\forall x,\ f_x(\varepsilon) \geq 0 = f_x(0)$" is false for all $\varepsilon > 0$, since $f_0(t) = -t^3$.*

PROPOSITION 11. *Let $\gamma(s) : \mathbf{R} \to K_1 \subset \mathbf{R}^n$ be a unit speed parametrization of a connected K_1 such that $\exp^\mu(\gamma(s), rN_\gamma(s)) = p_0$, $\forall s \in [s_1, s_2]$, for $s_1 < s_2$ and $r < UR(K_1, \mu)$ as in Proposition 10. Then, κ is a positive constant on the interval $[s_1, s_2]$ and*

$$(\mu')^2 - \mu\mu'' = \frac{1}{r_1^2} \text{ and } \gamma''' + \kappa^2\gamma' = 0,$$
$$\mu = \frac{2}{\kappa r_1} \cos\left(\frac{\kappa s}{2} + a\right) \text{ for some } a \in \mathbf{R}.$$

Therefore, Horizontal Collapsing Property occurs in a unique way only above arcs of circles of curvature κ and with specific μ. $\gamma([s_1, s_2]) \neq K_1$, even if $[s_1, s_2]$ is chosen to be a maximal interval satisfying above.

PROOF. By Propositions 8 and 10, $(\gamma(s), rN_\gamma(s)) \in Sng^{NK}(K, \mu)$ and

$$(\mu')^2 - \mu\mu'' = \frac{1}{r^2} \text{ and } \mu'' + \frac{1}{4}\kappa^2\mu = 0 \text{ with } \kappa > 0. \tag{6.2}$$

$$\begin{aligned}
0 &= \left((\mu')^2 - \mu\mu''\right)' = \left((\mu')^2 + \frac{1}{4}\kappa^2\mu^2\right)' \\
0 &= 2\mu'\mu'' + \frac{1}{2}\kappa\kappa'\mu^2 + \frac{1}{2}\kappa^2\mu\mu' \\
0 &= 2\mu'\left(\mu'' + \frac{1}{4}\kappa^2\mu\right) + \frac{1}{2}\kappa\kappa'\mu^2 \\
0 &= \frac{1}{2}\kappa\kappa'\mu^2
\end{aligned}$$

κ is constant, since κ and $\mu > 0$. $\mu = \frac{2}{\kappa r}\cos\left(\frac{\kappa s}{2} + a\right)$ is the only solution of (6.2).

$$\sqrt{1 - (r\mu')^2} = \frac{\kappa r \mu}{2} \text{ and } \gamma'' = \kappa N_\gamma$$

$$
\begin{aligned}
p_0 &= \exp^{\mu}(\gamma, rN_\gamma) = \gamma - r^2\mu\mu'\gamma' + r\mu\sqrt{1-(r\mu')^2}N_\gamma \\
0 &= \left(\gamma - r^2\mu\mu'\gamma' + \frac{1}{2}r^2\mu^2\gamma''\right)' \\
0 &= \left(1-(r\mu')^2 - r^2\mu\mu''\right)\gamma' + 0\cdot\gamma'' + \frac{1}{2}r^2\mu^2\gamma''' \\
0 &= \left(\frac{1}{4}\kappa^2\mu^2 - \mu\mu''\right)r^2\gamma' + \frac{1}{2}r^2\mu^2\gamma''' \\
0 &= \frac{1}{2}r^2\kappa^2\mu^2\gamma' + \frac{1}{2}\mu^2r^2\gamma''' = \frac{1}{2}\mu^2r^2\left(\kappa^2\gamma' + \gamma'''\right) \\
0 &= \kappa^2\gamma' + \gamma''' \\
p_1 &= \kappa^2\gamma + \gamma'' \text{ for some constant } p_1 \in \mathbf{R}^n \\
\left\|\frac{p_1}{\kappa^2} - \gamma\right\| &= \frac{1}{\kappa^2}\left\|\gamma''\right\| = \frac{1}{\kappa}
\end{aligned}
$$

γ is an arc of a circle in $\mathbf{R}^n$, since γ has curvature κ and lying on a sphere of radius $1/\kappa$, it has to be a great circle of that sphere. Since μ is not constant and K is compact, there are points where $\mu'' \geq 0$ on each component of K. However, on $[s_1, s_2]$, $\mu'' = -\frac{1}{4}\kappa^2\mu < 0$. $\gamma([s_1, s_2]) \neq K_1$. $\square$

Proposition 12. *Let $\{(K_i, \mu_i) : i = 1, 2, ...\}$ be a sequence where each K_i is a disjoint union of finitely many simple smooth closed curves in $\mathbf{R}^n$ with C^2 weight functions, and similarly for (K_0, μ_0). If $(K_i, \mu_i) \to (K_0, \mu_0)$ in C^2 topology, then*

$$\limsup_{i\to\infty} AIR(K_i, \mu_i) \leq AIR(K_0, \mu_0).$$

Proof. Let $\gamma_0(s) : domain(\gamma_0) \to K_0$ be a unit speed onto parametrization. Let $R > FocRad^-(K_0, \mu_0)$ be given arbitrarily. By Proposition 3, $\exists s_0 \in domain(\gamma_0)$ such that either $\Lambda(\kappa_0, \mu_0)(s_0)^{-\frac{1}{2}} < R$ with $\Delta(\kappa_0, \mu_0)(s_0) > 0$, or $\left|\mu_0'(s_0)\right|^{-1} < R$. By parametrizing all K_i over a small common open interval I about s_0 with respect to arclength, we can assume that $\mu_i'' \to \mu_0''$ and $\kappa_i \to \kappa_0$ uniformly on I. For sufficiently large i, $\Lambda(\kappa_i, \mu_i)(s_0)^{-\frac{1}{2}} < R$ with $\Delta(\kappa_i, \mu_i)(s_0) > 0$, or $\left|\mu_i'(s_0)\right|^{-1} < R$. Hence, $R > FocRad^-(K_i, \mu_i)$ for sufficiently large i.

$$\limsup_{i\to\infty} FocRad^-(K_i, \mu_i) \leq FocRad^-(K_0, \mu_0).$$

By Proposition 9, for all (K, μ) :

$$AIR(K, \mu) = UR(K, \mu) = \min\left(\frac{1}{2}DCSD(K, \mu), FocRad^-(K, \mu)\right).$$

Suppose that $\exists R_0$ such that $AIR(K_0, \mu_0) < R_0 < \limsup_{i\to\infty} AIR(K_i, \mu_i)$.

$$AIR(K_0, \mu_0) < R_0 < \limsup_{i\to\infty} FocRad^-(K_i, \mu_i) \leq FocRad^-(K_0, \mu_0) \tag{6.3}$$

$$AIR(K_0, \mu_0) = \frac{1}{2}DCSD(K_0, \mu_0) < R_0$$

$D(R_0) \subset W(\exp^{\mu_0}) \subset NK_0$ by (6.3). There exists a double critical pair (q_0, q_1) for (K_0, μ_0), and a point p on the line segment joining q_0 and q_1 such that $\|p - q_i\| = R_1\mu_0(q_i)$ and $p = \exp^{\mu_0}(q_i, R_1 v_i)$ with $v_i \in UN(K_0)_{q_i}$ for $i = 0, 1$ where $R_1 = AIR(K_0, \mu_0) < R_0$. As in the proof of Proposition 7(iii), we consider $\beta_1(s) =$

$\exp^{\mu_0}(q_1, sv_1)$ for $s \in (R_1, R_0)$. There exists at most one singular point along β_1 before R_0 by Proposition 2 and (6.3). By using Lemma 4 and the arguments in the proof of Proposition 7(iii) with $\measuredangle(\beta_1'(R_1), u(p, q_0)) = \alpha(q_1, p) - \frac{\pi}{2} < \frac{\pi}{2}$, choose $s_1 \in (R_1, R_0)$ such that $\|\beta_1(s_1) - q_0\| \, \mu_0(q_0)^{-1} < R_1$ and $\exp^{\mu_0}$ is not singular at (q_1, s_1v_1). There exists an open connected set $V_1^T \subset D(R_0) - D(R_1) \subset NK_0$ such that

i. $(q_1, s_1v_1) \in V_1^T$,

ii. $\exp^{\mu_0} \mid V_1^T$ is a diffeomorphism onto an open set V_1 ($\subset \mathbf{R}^n$) containing $\beta_1(s_1)$,

iii. $0 < c_1 \le \inf \left\| d(\exp^{\mu_0} \mid V_1^T) \right\| \le \sup \left\| d(\exp^{\mu_0} \mid V_1^T) \right\| \le C_1 < \infty$,

iv. $\|x - q_0\| \, \mu_0(q_0)^{-1} < R_1$, $\forall x \in V_1$, and

v. $\{q \in K_0 : (q, w) \in V_1^T\}$ is an open arc whose length is much shorter than the length of the component of K_0 containing q_1.

There exists a μ_0-closest point $q_2 \in K_0$ to $\beta_1(s_1)$, and $\beta_1(s_1) = \exp^{\mu_0}(q_2, R_2v_2)$ where $R_2 < R_1$. By Proposition 1(ii, v), $q_1 \ne q_2$, since $R_1 < |\mu'(q_1)|^{-1}$. Let $\beta_2(s) = \exp^{\mu_0}(q_2, sv_2)$. There exists $s_2 < R_2$ sufficiently close to R_2 such that $\exp^{\mu_0}$ is not singular at (q_2, s_2v_2) and $\exp^{\mu_0}(q_2, s_2v_2) \in V_1$. There exists an open set $V_2^T \subset D(R_2) \subset NK_0$ such that $(q_2, s_2v_2) \in V_2^T$, $\exp^{\mu_0} \mid V_2^T$ is a diffeomorphism onto an open set V_2 with $\beta_2(s_2) \in V_2 \subset V_1$, and satisfying the same type conditions as (iii) and (v) above. $V_1^T \cap V_2^T \subset V_1^T \cap D(R_2) = \varnothing$.

Let K_0' be open subset of K_0 such that $V_1^T \cup V_2^T \subset NK_0'$. Having chosen V_i^T small, we can assume that K_0' is a union of one or two short open arcs, neither of which is a whole component of K_0. Parametrize $\gamma_0 : I_0 \to K_0'$ and for sufficiently large $i \ge i_0$, $\gamma_i : I_0 \to K_i' \subset K_i$ with unit speed s so that $\{\gamma_i|I_0\}_{i=i_0}^{\infty}$ converges to $\gamma_0|I_0$ uniformly in C^2 topology as $i \to \infty$. All NK_i' are diffeomorphic to (and can be identified with) the fixed NK_0'. Since $(K_i, \mu_i) \to (K_0, \mu_0)$ in C^2 topology, $\exp^{(K_i', \mu_i)} : NK_i' \simeq NK_0' \to \mathbf{R}^n$ converges to $\exp^{(K_0', \mu_0)}$ in C^1 topology. $V_1^T \cap V_2^T = \varnothing$, but $\exp^{(K_0', \mu_0)}(V_2^T) \subset \exp^{(K_0', \mu_0)}(V_1^T)$ where all are open sets, and $\exp^{(K_0', \mu_0)}$ is a local diffeomorphism on $V_1^T \cup V_2^T$ satisfying (iii). Therefore, for sufficiently large i, $\exp^{(K_i', \mu_i)}$ is a local diffeomorphism on $V_1^T \cup V_2^T \subset D(R_0)$ where V_1^T and V_2^T are nonempty disjoint open sets, but $\exp^{(K_i', \mu_i)}(V_2^T) \cap \exp^{(K_i', \mu_i)}(V_1^T) \ne \varnothing$. Therefore, by the definition, $AIR(K_i, \mu_i) \le R_0$ for sufficiently large i. This contradicts with the conditions of the initial choice of R_0. The nonexistence of such R_0 proves that $\limsup_{i \to \infty} AIR(K_i, \mu_i) \le AIR(K_0, \mu_0)$. □

Proof. **Theorem 3** Assume that $R = TIR(K, \mu) < UR(K, \mu)$. Recall the proof of Proposition 4(i) that (i) $\exp^{\mu} : D(R) \to O(K, \mu R)$ is a homeomorphism, and $\forall R'$ such that $R < R' < UR(K, \mu)$, $\exp^{\mu} \mid D(R')$ is not injective. By Proposition 10(iii, iv), there exists $p_0 = \exp^{\mu}(\gamma(s), rN_{\gamma}(s)) \in Sng(K, \mu)$ for some parametrization γ of K, $\forall s \in [s_1, s_2]$ for some $s_1 < s_2$, and $R \le r < R'$. By Proposition 11, $\gamma([s_1, s_2])$ is a desired arc of a circle with compatible μ. Conversely, if such an arc of a circle exists, with compatible μ, then as it was discussed in Example 1, there exists a horizontal collapsing curve $\exp^{\mu}(\gamma(s), r'N_{\gamma}(s)) = p_0'$ with $\forall s \in [s_1', s_2']$ for some $s_1' < s_2'$, which must satisfy $R \le r'$. Therefore, $TIR(K, \mu)$ is equal to the infimum of such r. If the lengths of disjoint collapsing curves converges to zero and their μ-height decreases to R, then it is possible that the infimum may not be attainable. If there are no such circles, then $\exp^{\mu} : D(UR) \to O(K, \mu UR)$ is injective, and hence it is a homeomorphism by repeating the proof of Proposition 4(i). □

The proof of **Theorem 1** is provided by Propositions 4, 5, 7, 9, 12, and Lemma 6. The proof of **Theorem 2** is provided by Propositions 6, 10 and 11. The proof of **Theorem 4** is provided by Propositions 8, 9 and 10.

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