LOCAL STRUCTURE OF IDEAL SHAPES OF KNOTS

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ABSTRACT. Relatively extremal knots are the relative minima of the ropelength functional in the C^1 topology. They are the relative maxima of the thickness (normal injectivity radius) functional on the set of curves of fixed length, and they include the ideal knots. We prove that a $C^{1,1}$ relatively extremal knot in $\mathbf{R}^n$ either has constant maximal (generalized) curvature, or its thickness is equal to half of the double critical self distance. This local result also applies to the links. Our main approach is to show that the shortest curves with bounded curvature and C^1 boundary conditions in $\mathbf{R}^n$ contain CLC (circle-line-circle) curves, if they do not have constant maximal curvature.

1. INTRODUCTION

In this article, we study the local structure of $C^{1,1}$ relatively extremal knots in $\mathbf{R}^n$ ($n \geq 2$), by using a length minimization problem with bounded curvature and C^1 boundary conditions. The thickness of a knotted curve is the radius of the largest tubular neighborhood around the curve without intersections of the normal discs. This is also known as the normal injectivity radius $NIR(K, \mathbf{R}^n)$ of the normal exponential map of the curve K in the Euclidean space $\mathbf{R}^n$. The ideal knots are the embeddings of S^1 into $\mathbf{R}^n$, maximizing $NIR(K, \mathbf{R}^n)$ in a fixed isotopy (knot) class of fixed length. A relatively extremal knot is a relative maximum of $NIR(K, \mathbf{R}^n)$ in the C^1 topology, if the length is fixed.

Several different notations for thickness appeared in the literature. $R(K)$ was used for thickness in [LSDR] and [BS]. [GM] showed that the thickness $\eta_*(K)$ was equal to the minimum $\Delta(K)$ of ρ_G, the global radius of curvature for C^2 curves. In [CKS], Cantarella-Kusner-Sullivan defined thickness $\tau(K)$ by the infimum of the global radius of curvature and proved that it was the normal injectivity radius for $C^{1,1}$ curves. They also defined ideal (thickest) knots and links as "*tight*".

We prove every result in $\mathbf{R}^n$ ($n \geq 2$) in this article, since our methods are not dependent on the dimension. Although ideal knots are not interesting for $n \neq 3$ since they are trivial, relatively extremal knots, the length minimization with bounded curvature, and some of the local results on curves we obtained may be useful for other purposes.

As noted in [Ka], "...the average shape of knotted polymeric chains in thermal equilibrium is closely related to the ideal representation of the corresponding knot type". "Knotted DNA molecules placed in certain solutions follow paths of random

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closed walks and the ideal trajectories are good predictors of time averaged properties of knotted polymers" as a biologist referee pointed out to the author. Since the length of the molecule is fixed, this problem becomes the maximization of its thickness within a fixed isotopy class of a knot. The analytical properties of the ideal knots will be useful tools in the research on the knotted polymers.

Approximations of the ideal shapes of some simple knots have been obtained using computers, see [Ka], [GM] and [SKK]. Gonzales and Maddocks used the notion of the global radius of curvature on discrete curves to obtain approximations of the ideal shapes in [GM]. However, we do not know the exact shape of the most of the nontrivial knots (including trefoil knots) in ideal position or the exact value of their thickness. Some estimates of the thickness of ideal knots have been obtained by Litherland et al. [LSDR], Cantarella, Kusner and Sullivan [CKS], Diao [Di], Buck and Simon [BS], and Rawdon and Simon [RS] by using results of Freedman, He and Wang [FHW].

One of the earliest results about the shape of ideal knots was obtained by Gonzales and Maddocks [GM, p 4771]: *A smooth ideal knot can be partitioned into arcs of constant (maximal) global curvature and line segments.*

Since a positive lower bound on thickness bounds curvature, the completion of this class must include $C^{1,1}$ curves. The extremal cases in $\mathbf{R}^3$ are unlikely to be smooth. Very few ideal knots in $\mathbf{R}^3$ are expected to be C^2, and the unknotted standard circles are possibly the only smooth ones. [CKS] discovered examples of tight (ideal) links which are $C^{1,1}$ but not C^2. Hence, we will study $NIR(K, \mathbf{R}^n)$ for the $C^{1,1}$ curves. One of our results, *Proposition 8*, generalizes [GM]'s result in the preceding paragraph to the unions of finitely many disjoint $C^{1,1}$ simple closed curves which are relative minima of ropelength.

In [D1], the author proved the following Thickness Formula in the general context and developed the notion of "Geometric Focal Distance, $F_g(K)$" by using metric balls, which naturally extends the notion of the focal distance of the smooth category to the C^1 category. $DCSD(K)$ is the double critical self distance and $R_O(K, M)$ is the rolling ball radius. See Section 2 for the basic definitions.

GENERAL THICKNESS FORMULA *[D1, Theorem 1]*

For every complete smooth Riemannian manifold M^n and every compact $C^{1,1}$ submanifold K^k ($\partial K = \emptyset$) of M,

$$NIR(K, M) = R_O(K, M) = \min\{F_g(K), \frac{1}{2}DCSD(K)\}.$$

Nabutovsky, [N] had an extensive study of $C^{1,1}$ hypersurfaces K in $\mathbf{R}^n$ and their injectivity radii. [N] proved the upper semicontinuity of $NIR(K, \mathbf{R}^n)$ and lower semicontinuity of $vol(K)/NIR(K, \mathbf{R}^n)^k$ in C^1 topology. These were also done by Litherland in [L] for $C^{1,1}$-knots in $\mathbf{R}^3$ and by [CKS] for links. We will use *Corollary 1* of the formula for curves in $\mathbf{R}^n$ in Section 4. $NIR(K, M) = R_O(K, M)$, a rolling ball/bead description of the injectivity radius in $\mathbf{R}^n$, was known by Nabutovsky for hypersurfaces, by Buck and Simon for C^2 curves, [BS], and by Cantarella, Kusner and Sullivan [CKS], Lemma 1. The rolling ball/bead characterization is our main geometric tool.

For a $C^{1,1}$ curve γ, γ'' exists almost everywhere by Rademacher's Theorem, [F]. For a $C^{1,1}$ curve $\gamma(s)$ parametrized by the arclength s, define the (generalized) curvature $\kappa\gamma(s) = \limsup\limits_{x\neq y\to s}\frac{\measuredangle(\gamma'(x),\gamma'(y))}{|x-y|}$ for all s and the analytic focal distance $F_k(\gamma) = (\sup\ \kappa\gamma)^{-1}$. For K with several components, take the smallest focal distance of the components. See *Lemmas 1* and *2*, for a proof of $F_g(\gamma) = F_k(\gamma) = (ess\sup\|\gamma''\|)^{-1}$ for curves parametrized by arclength in $\mathbf{R}^n$, and [D1, Proposition 12] for a similar curvature description of $F_g(K^k)$ for higher dimensional $K^k \subset \mathbf{R}^n$.

Corollary 1. *(Thickness Formula) For every union K of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$, one has*
$NIR(K,M) = R_O(K,M) = \min\{F_k(K), \frac{1}{2}DCSD(K)\}$.

Thickness Formula was first discussed for C^2-knots in $\mathbf{R}^3$ in [LSDR], and for $C^{1,1}$ knots in $\mathbf{R}^3$ by Litherland in [L], without R_O. [CKS, Lemma 1] proved the Thickness Formula for $C^{1,1}$ knots and links in $\mathbf{R}^3$, since $F_g = F_k$.

The notion of the global radius of curvature ρ_G was developed by Gonzales and Maddocks in [GM] for smooth curves in $\mathbf{R}^3$, defined by using circles passing through triple distinct points of the curve. This gives another characterization: $NIR(K,\mathbf{R}^3) = \inf_K \rho_G$, [GM]. This is still true for all C^1 curves by [CKS, Lemma 1]. The construction of ρ_G and R_O for curves in $\mathbf{R}^3$ are different in nature due to 3-point intersection condition versus 1-point of tangency and 1-point of intersection condition. However, at their infima they are the same quantity: $NIR(K,\mathbf{R}^3)$, [CKS, Lemma 1]. Although the equality $NIR(K,M) = R_O(K,M)$ is generalizable to all dimensions and Riemannian manifolds [D1], the notion of ρ_G may not be effective beyond the spaces of constant curvature.

Remark 1. *In this article, we study the pieces of relatively extremal knots away from the points of minimal double critical self distance by using minimization of length with curvature bounded by Λ. Since all problems we discuss are in $\mathbf{R}^n$, we can rescale and take $\Lambda = 1$ to simplify our statements and proofs.*

Problem 1. *(Markov [M]-Dubins [Du]) Given p, q, v, w in $\mathbf{R}^n$, with $\|v\| = \|w\| = 1$. Classify all shortest curves in $\mathcal{C}(p,q;v,w)$ which is the set of all curves γ between the points p and q in $\mathbf{R}^n$ with $\gamma'(p) = v$, $\gamma'(q) = w$ and $\kappa\gamma \leq 1 = \Lambda$.*

A *CLC(circle-line-circle)* curve is one circular arc followed by a line segment and then by another circular arc in a C^1 fashion (like two letters J with common straight parts, one hook at each end, and possibly non-coplanar, see *Figure 10*), where the circular arcs have radius 1. We will use *CLC(Λ)* if the upper bound of curvature is Λ. Similarly, one can define CCC-curves by C^1-concatenation of 3 arcs of circles of the same curvature. If $p = q$ and $v = -w$, then the shortest curve with curvature restriction satisfies $\kappa \equiv 1$ and it is not a CLC-curve, see the examples and remarks following the proof of *Theorem 1* as well as *Figure 11.* One can construct curves of constant (generalized) curvature 1 such that the curve is not twice differentiable at countably infinite points.

We note that the classification of shortest curves in $\mathcal{C}(p,q;v,w)$ with $\kappa \equiv 1$ is not a simple matter. Markov [M], Dubins [Du] and Reeds and Shepp [ReSh] studied the 2 dimensional cases. In dimension 3, the following results of H. Sussmann obtained the possible types solutions for this problem. A helicoidal arc is a smooth curve in $\mathbf{R}^3$ with constant curvature 1 and positive torsion τ satisfying the differential equation $\tau'' = 1.5\tau'\tau^{-1} - 2\tau^3 + 2\tau - \zeta\tau\,|\tau|^{1/2}$ for some nonnegative constant ζ.

THEOREM. *(Sussmann [S])*

1. For the Markov-Dubins problem in dimension three, every minimizer is either (a) a helicoidal arc or (b) a concatenation of three pieces each of which is a circle or a straight line. For a minimizer of the form CCC, the middle arc has length $\geq \pi$ *and* $< 2\pi$.

2. Every helicoidal arc corresponding to a value of ζ *such that* $\zeta > 0$ *is local strict minimizer.*

Sussmann further proved that *CSC Conjecture* (every minimizer is either *CCC* or *CLC,* [ReSh]) is false in $\mathbf{R}^3$ [S, Proposition 2.1 and 2.2]. In [S], the proofs of Propositions 2.1 and 2.2, a detailed outline of the main steps of the proof of Sussmann's Theorem 1, and some remarks about that of Sussmann's Theorem 2 are given. For a given initial data in $\mathbf{R}^3$, the determination of which type minimizing curve, τ and ζ still needs to be studied. The complete set of types of minimizing curves is not known in dimensions $n \geq 4$.

Theorem 1 of this article below provides some answers in $\mathbf{R}^n$ except the constant maximal curvature case. The methods used by Sussmann were in Control theory, ordinary differential equations, and were specific to dimension 3. In contrast, the results of our article are proved by using simple geometric methods in $\mathbf{R}^n$, and the proofs are independent of [S].

Theorem 1. *Let* $\gamma : I = [0, L] \to \mathbf{R}^n$ *be a shortest curve in* $\mathcal{C}(p, q; v, w)$ *parametrized by arclength.*

(a) If γ *does not have constant curvature* 1*, i.e.* $\kappa\gamma(s_0) < 1$ *for some* s_0*, then there exist* a *and* b *such that* $s_0 \in [a, b] \subset [0, L]$ *and* $\gamma([a, b])$ *is a* CLC*-curve where the line segment of positive length contains* $\gamma(s_0)$ *and each circular part has length at least* π *unless it contains the initial or the terminal point of* γ.

(b) If $R_O(\gamma(I), \mathbf{R}^n) \geq 1$ *and* γ *does not have constant curvature* 1*, then all of* γ *is a* CLC *curve where each circular part has length at most* π *and the line segment has positive length.*

Theorem 1 tells us that the parts of a relatively extremal knot with the points of minimal double critical self distance removed are CLC-curves or overwound, i.e. $\kappa \equiv 1$. As J. Simon pointed out that there are physical examples (no proofs) of relatively extremal unknots in $\mathbf{R}^3$, which are not circles, and hence not ideal knots. One can construct similar physical examples for composite knots.

Given a certain type of knot and a rope of set thickness, finding the exact shape to tie the knot by using the shortest amount of the rope is basically the same as finding the shape of a thickest knot of fixed length in this knot type in $\mathbf{R}^3$. For any (finite union of finitely many disjoint) $C^{1,1}$ simple closed curve(s) γ in $\mathbf{R}^n$, define the ropelength or extrinsically isoembolic length to be $\ell_e(\gamma) = \frac{\ell(\gamma)}{R_o(\gamma)}$ where $\ell(\gamma)$ is the length of γ. The notion of ropelength has been defined and studied by several authors, for example, Litherland-Simon-Durumeric-Rawdon [LSDR] (called its reciprocal thickness), Gonzales and Maddocks [GM], Cantarella-Kusner-Sullivan [CKS], Litherland [B], Buck-Simon [BS]. A curve γ_0 is called an ideal (thickest or tight) knot or link in a class $[\theta]$, if ℓ_e attains its absolute minimum over $[\theta]$ at γ_0; and γ_0 is called relatively extremal, if ℓ_e attains a relative minimum at γ_0 with respect to C^1 topology. Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$ and $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ be a parametrization. We define

$I_c(K) = \{x \in \mathbf{D} : \exists y \in \mathbf{D} - \{x\}$ such that $\|\gamma(x) - \gamma(y)\| = DCSD(K)$ and $\gamma(x) - \gamma(y) \perp K$ at both $\gamma(x)$ and $\gamma(y)\}$.

Theorem 2. *Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$ and $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ be a parametrization. If γ is a relative minimum for ℓ_e and $\exists s_0 \in \mathbf{D}, \kappa\gamma(s_0) < \sup \kappa\gamma$, then both of the following hold for $\gamma(\mathbf{D}) = K$:*

(a) $NIR(K, \mathbf{R}^n) = R_O(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$.

(b) If $s_0 \notin I_c(K)$, then there exists a, b such that $s_0 \in [a, b]$, $\gamma([a, b])$ is a CLC$(\sup \kappa\gamma)$*-curve where the line segment has positive length and contains $\gamma(s_0)$, and each circular part has at most π radians angle ending at a point of $I_c(K)$.*

As an immediate consequence we obtain the following.

Corollary 2. *Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$. If K is a relative minimum for ℓ_e and curvature of K is not identically constant $R_O(K)^{-1}$, then the thickness of K is $\frac{1}{2}DCSD(K)$.*

Equivalently, if there exists a relative minimum K for ℓ_e such that $\frac{1}{2}DCSD(K) > R_O(K) = F_k(K)$, then K must have constant generalized curvature $F_k(K)^{-1}$.

Remark 2. *In dimension $n = 3$, there does not exist a link or a knot in $\mathbf{R}^3$ with constant generalized curvature $\kappa\gamma \equiv R_O(K)^{-1}$ and $R_O(K) = F_k(\gamma) < \frac{1}{2}DCSD(K)$, by the results of our forthcoming article:*

THEOREM. *[D2, Theorem 1] Let n be a dimension such that*

(i) every minimizer for the Markov-Dubins problem in $\mathbf{R}^n$ is either a smooth curve with curvature 1 and positive torsion, or a C^1-concatenation of finitely many circular arcs of curvature 1 *and a line segment, and*

(ii) every $CCC-$curve with the middle arc of length $< \pi$ is not a minimizer.

Then, $NIR(K, \mathbf{R}^n) = \frac{1}{2}DCSD(K)$ for every relative minimum K of ℓ_e where K is a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$.

COROLLARY *[D2, Corollary 1] Let K be a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^3$. If K is a relative minimum of ℓ_e, then*

$NIR(K, \mathbf{R}^3) = \frac{1}{2}DCSD(K)$.

Proposition 8 and [GM, section 4 for smooth knots] obtain that ideal knots away from the maxima of the global radius of curvature ρ_G consists of line segments. However, maximal ρ_G does not distinguish between the points of minimal double critical self distance and points of maximal curvature. For an ideal (or relatively extremal) $C^{1,1}$ knot or link, *Theorem 2* proves that (i) after a line segment, the ideal curve must go through a minimal double critical point before reaching the next line segment, and (ii) if there is a non-linear piece of the ideal curve(s) between a line segment and the next minimal double critical point of the same component, then that must be a planar circular arc whose radius is the thickness of the ideal knot.

Basic definitions are given in Section 2, shortest curves with curvature restrictions and proof of *Theorem 1* are given in Section 3, and ideal knots and proof of *Theorem 2* are given in Section 4.

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2. Basic definitions for Thickness Formula

Let K denote a union of finitely many disjoint simple C^1 curves in $\mathbf{R}^n$ throughout this section. For the generalizations of the following concepts to $C^{1,1}$ submanifolds of Riemannian manifolds, we refer to [D1].

Definition 1. *TK and NK denote the tangent and normal bundles of the submanifold K in $\mathbf{R}^n$, respectively. UTK and UNK denote the unit vectors, NK_p denotes the set normal vectors at p, and similarly for the others.*

Definition 2. *For a metric space (X,d), $B(p,r)$ and $\bar{B}(p,r)$ denote open and closed metric balls. For $A \subset X$, $B(A,r) = \{x \in X : d(x,A) < r\}$. The diameter $d(X)$ is $\sup\{d(x,y) : x,y \in X\}$. If there is an ambiguity, we will use d_X and $B(p,r;X)$. For a curve $\gamma \subset A \subset \mathbf{R}^n$, $\ell(\gamma)$ is the length of γ in $\mathbf{R}^n$. $\ell_{ab}(\gamma)$ and $\ell_{pq}(\gamma)$ both denote the length of γ between $\gamma(a) = p$ and $\gamma(b) = q$.*

Definition 3. $\exp_p^N v = p + v : NK \to \mathbf{R}^n$ *is the normal exponential map of K in $\mathbf{R}^n$. The thickness of K in $\mathbf{R}^n$ or the normal injectivity radius of* $\exp^N$ *is*

$$NIR(K,\mathbf{R}^n) = \sup(\{0\} \cup \{r > 0 : \exp^N : \{v \in NK : \|v\| < r\} \to M \text{ is one-to-one}\}).$$

Equivalently, if $\gamma(s)$ parametrizes K, then

$$r > NIR(K,\mathbf{R}^n) \Leftrightarrow \left(\begin{array}{c} \exists \gamma(s), \gamma(t), q \in \mathbf{R}^n, \\ \gamma(s) \neq \gamma(t), \|\gamma(s) - q\| < r, \|\gamma(t) - q\| < r, \text{ and} \\ (\gamma(s) - q) \cdot \gamma'(s) = (\gamma(t) - q) \cdot \gamma'(t) = 0 \end{array} \right).$$

Figure 1 shows the smoothness of the boundary of a tubular neighborhood when $r < NIR(K,\mathbf{R}^n)$. *Figure 2* shows the failure of injectivity near p, q, x and y when $r > NIR(K,\mathbf{R}^n)$. *Figure 3* is a magnification of *Figure 2* near p, showing the singular behavior near focal points.

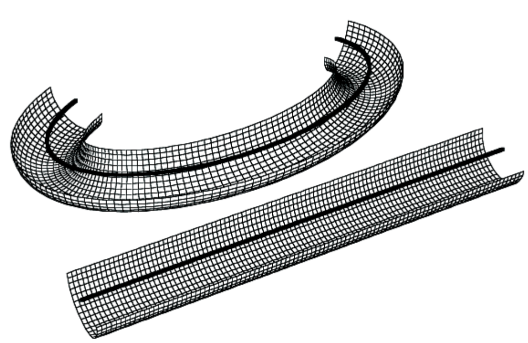

Figure 1. The case of $r < NIR(K)$: A portion of $\exp^N(\{v : \|v\| = r\})$ which is the smooth boundary of the tubular $r-$neighborhood

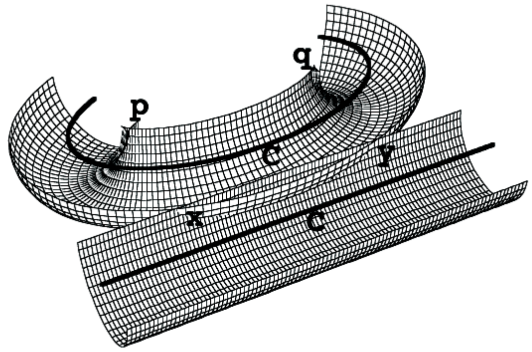

Figure 2. The case of $r > NIR(K)$:
$\exp^N(\{v : \|v\| = r\})$ is not smooth near p and q, and the failure of injectivity can be seen around the points p, q, x and y.

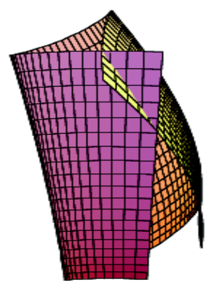

Figure 3. A magnification of Figure 2 near p

Definition 4. *For any $v \in UT\mathbf{R}^n_p$ and any $r > 0$, define*

(a) $O_p(v,r) = \bigcup_{w \in v^\perp(1)} B(\exp_p rw, r)$, *where* $v^\perp(1) = \{w \in UT\mathbf{R}^n_p : \langle v, w\rangle = 0\}$

or equivalently,

$$O_p(v,r) = \{x \in \mathbf{R}^n : \exists w \in \mathbf{R}^n, v \cdot w = 0, \|w\| = 1, \|x - p - rw\| < r\}$$

(b) $O^c_p(v,r) = \mathbf{R}^n - O_p(v,r)$

(c) $O_p(r;K) = O_p(v,r)$ *where* $v \in UTK_p$

(d) $O(r;K) = \bigcup_{p \in K} O_p(r;K)$

In all of the above, r may be omitted when $r = 1$. K will be omitted unless there is ambiguity.

Intuitively, $\partial O_p(v,r)$ is a torus pinched at p in $\mathbf{R}^3$, see *Figure 4*. $p \notin O_p(v,r)$ which is the the open set inside the pinched torus, referred by some as a "fat torus". $\overline{O_p(v,r)}$ is a donut pinched at p. *Figure 5* shows a portion of $\partial O_p(v,r)$ and the behavior around p.

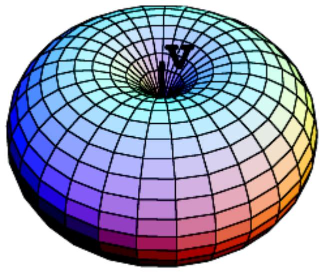

Figure 4. The pinched torus bounding $O_p(v)$

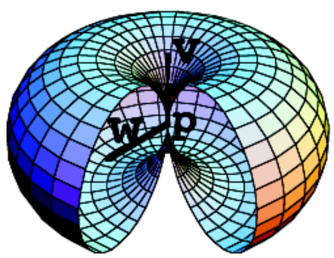

Figure 5. A portion of $\partial O_p(v)$

Definition 5. *(a) The ball radius of K in $\mathbf{R}^n$ is*

$$R_O(K,\mathbf{R}^n) = \inf\{r > 0 : O(r;K) \cap K \neq \emptyset\}.$$

(b) The pointwise geometric focal distance at $p \in K$ is

$$F_g(p) = \inf\{r > 0 : p \in \overline{O_p(r;K) \cap K}\}.$$

and the geometric focal distance of K is $F_g(K) = \inf_{p\in K} F_g(p)$. Of course, on a line segment one has $F_g(p) = \infty = \inf \varnothing$.

Definition 6. *A pair of distinct points p and q in K are called a double critical pair for K, if the line segment $\overline{pq}$ is normal to K at both p and q. The double critical self distance is*

$$DCSD(K) = \inf\{\|p-q\| : \{p,q\} \text{ is a double critical pair for } K\}$$

A double critical pair $\{p,q\}$ is called minimal if $DCSD(K) = \|p-q\|$. We denote such a pair as MDC.

3. Shortest Curves in $\mathbf{R}^n$ with Curvature Restrictions

In this section, $\gamma : I \to \mathbf{R}^n$ denotes a simple C^1 curve with a compact interval I, $\|\gamma'\| \neq 0$ and $K = image(\gamma)$, unless stated otherwise**.** $\{e_i : i = 1,2,..,n\}$, E_i, E_i^+, E_i^- denote the standard basis in $\mathbf{R}^n$, the $e_i - axis$, its positive and negative parts, respectively.

Definition 7. *For $\gamma : I \to \mathbf{R}^n$, define:*

Dilations: $dil^d\gamma'(s,t) = \frac{\|\gamma'(s)-\gamma'(t)\|}{\ell(\gamma([s,t]))}$ and $dil^\alpha\gamma'(s,t) = \frac{\measuredangle(\gamma'(s),\gamma'(t))}{\ell(\gamma([s,t]))}$ for $s \neq t$

(Generalized) Curvature: $\kappa\gamma(s) = \limsup_{t\neq u \text{ and } t,u\to s} dil^\alpha\gamma'(t,u)$

Lower curvature: $\kappa^{-}\gamma(s) = \limsup_{t \to s} \; dil^{\alpha}\gamma'(s,t)$

Analytic focal distance: $F_k(\gamma) = (\sup_I \kappa\gamma(s))^{-1}$. *If* K *is a union of finitely many disjoint* $C^{1,1}$ *curves* $\gamma_{(i)}$ *in* $\mathbf{R}^n$, *then* $F_k(K) = \min_i F_k(\gamma_{(i)})$.

Remark 3. *(a) Since* $\lim_{v \to w} \frac{\measuredangle(v,w)}{\|v-w\|} = 1$ *for* $\|v\| = \|w\| = 1$, *one obtains the same* $\kappa\gamma$, *if one uses* dil^d *instead of* dil^{α}, *provided that* $\|\gamma'\| \equiv 1$. *The same is true for* $\kappa^{-}\gamma$.

(b) $\kappa\gamma(s) \geq \kappa^{-}\gamma(s), \forall s$.

(c) If $\gamma \in C^{1,1}$ *and* $\|\gamma'\| \equiv 1$, *then* $\|\gamma''(s)\| = \kappa^{-}\gamma(s)$, *for almost all* s.

(d) $\limsup_{s_n \to s} \; \kappa\gamma(s_n) \leq \kappa\gamma(s)$.

Lemma 1. *All of the following are equivalent for* $\gamma : I \to \mathbf{R}^n$ *with* $\|\gamma'\| \equiv 1$.

(a) $\kappa\gamma(s) \leq 1, \forall s \in I$.

(b) $dil^d\gamma'(s,t) \leq 1, \forall s,t \in I$.

(c) $dil^{\alpha}\gamma'(s,t) \leq 1, \forall s,t \in I$.

(d) $\|\gamma''(s)\| \leq 1$ *for almost all* $s \in I$, *and* γ' *is absolutely continuous.*

Proof. $(a \Longrightarrow c) : \forall s < t < u, dil^{\alpha}\gamma'(s,u) \leq \max(dil^{\alpha}\gamma'(s,t), dil^{\alpha}\gamma'(t,u))$. Hence, if $dil^{\alpha}\gamma'(s,u) \geq A$ for some $s \neq u$ and A, then there exists $s_0 \in [s,u]$ with $\kappa\gamma(s_0) \geq A$. $(d \Longrightarrow a) : \|\gamma'(t) - \gamma'(s)\| = \left\| \int_s^t \gamma''(u)du \right\| \leq \int_s^t \|\gamma''(u)\| \, du \leq \|t-s\|$, by absolute continuity. The rest are obvious. □

Definition 8. *A* $C^{1,1}$ *curve* $\gamma : I = [a,b] \to \mathbf{R}^n$ *is called a* $CLC-$*curve if there exists* $[c,d] \subset [a,b]$ *such that (a)* $\gamma([c,d])$ *is a line segment of possibly zero length, and (b) each of* $\gamma([a,c])$ *and* $\gamma([d,b])$ *is a planar circular arc of radius* 1 *and of length in* $[0,2\pi)$. *The curve* γ *need not be planar. We will use* CLC(Λ), *if the upper bound of curvature is* Λ.

Proposition 1. *Let* $p \in \mathbf{R}^n$ *and* $v \in \mathbf{R}^n$ *with* $\|v\| = 1$.

(a) $\forall q \in O_p^c(v), \exists w \in \mathbf{R}^n, \exists q' \in \partial O_p^c(v)$ *and a* C^1 *curve* $\gamma_{pq} \subset \{p\} + span\{v,w\}$ *such that*

(a.i) $v \cdot w = 0$, $\|w\| = 1$, *and*

(a.ii) $\gamma_{pq}(t) = \begin{cases} p + v\sin t + w(1-\cos t) & \text{if } 0 \leq t \leq t_0 \\ q' + (t-t_0)(q-q')/\|q-q'\| & \text{if } t_0 \leq t \leq t_1 = t_0 + \|q-q'\| \end{cases}$,

where $q' = \gamma_{pq}(t_0)$ *and* $q = \gamma_{pq}(t_1)$, *and*

(a.iii) γ_{pq} *is a shortest curve among all the continuous curves* φ *from* p *to* q *in* $O_p^c(v)$ *with* $(\varphi(t) - p) \cdot v > 0$ *for small* $t > 0$ *and* $\varphi(0) = p$.

(b) If $q - p \neq \lambda v, \forall \lambda \in \mathbf{R}$, *then* q' *and* w *are unique and* γ_{pq} *is unique up to parametrization. Consequently,* γ_{pq} *depends on* p, q *and* v *continuously, if* $q - p \neq \lambda v, \forall \lambda$.

See *Figure 6* for *Proposition 1.*

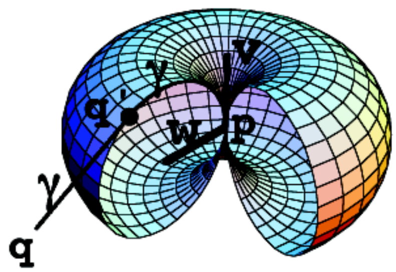

Figure 6. $\gamma_p q$ of Proposition 1

Proof. It suffices to prove all for $p = \mathbf{0}$, $v = e_1$ and $q \neq \mathbf{0}$, using $\gamma = \gamma_{pq}$. Consider the non-empty set of all rectifiable curves φ of length $\leq L$ satisfying (a.iii) for some sufficiently large $L < \infty$. Parametrize each curve φ by arclength and extend its domain to $[0, L]$ by keeping φ constant after reaching q so that $\|\varphi(s) - \varphi(t)\| \leq |s-t|, \forall s, t$. This forms a non-empty, bounded and equicontinuous family, and length functional is lower semi-continuous under uniform convergence. By Arzela-Ascoli Theorem, a shortest $C^{0,1}$ curve γ from $\mathbf{0}$ to q in $O^c_{\mathbf{0}}(e_1)$ satisfying (a.iii) exists. The proof below will also show that we can deform any curve φ as in (a.iii) to a shorter curve until we reach γ which can be reparametrized to be C^1. For any $u \in \mathbf{R}^n$, define $u^N = u - (u \cdot e_1)e_1$.

Case 1. If $q \in E_1^+$, then γ is the line segment from $\mathbf{0}$ to q where $q' = \mathbf{0}$ and $t_0 = 0$. Furthermore, if γ intersects E_1^+ at any $q'' \neq \mathbf{0}$, then $q \in E_1^+$. For, γ must be along E_1^+ between $\mathbf{0}$ and q'', and then it extends uniquely as a geodesic of $\mathbf{R}^n$ beyond $q'' \in int\ O^c_{\mathbf{0}}(e_1)$ to q.

Case 2. $\gamma \cap E_1 = \{\mathbf{0}\}$. Let $w = q^N / \|q^N\|$, since $q^N \neq \mathbf{0}$. It suffices to prove the rest for $w = e_2$. Define

$$f : \mathbf{R}^n - E_1 \to A = \{xe_1 + ye_2 : x, y \in \mathbf{R} \text{ and } y > 0\} \text{ by } f(u) = (u \cdot e_1)e_1 + \|u^N\| e_2.$$

f is a smooth length decreasing map:

$$\begin{aligned}\|f(u) - f(z)\|^2 &= \|(u \cdot e_1)e_1 - (z \cdot e_1)e_1\|^2 + \left(\|u^N\| - \|z^N\|\right)^2 \\ &\leq \|(u - z) \cdot e_1\|^2 + \|u^N - z^N\|^2 = \|u - z\|^2 .\end{aligned}$$

Equality holds if and only if $u^N = cz^N$ for some $c > 0$, implying $u \in span(e_1, z)$. Parametrize γ with respect to arclength. Extend $f \circ \gamma$ by $f(\gamma(0)) = \mathbf{0}$. By following Federer [F], pp. 109, 163-168, we obtain that γ and $f \circ \gamma$ are lipschitz, absolutely continuous, γ' and $(f \circ \gamma)'$ exist a.e., and

$$\ell(\gamma) = \int_0^{\ell(\gamma)} \|\gamma'(s)\|\, ds \geq \int_0^{\ell(\gamma)} \|f_* \gamma'(s)\|\, ds \geq \ell(f(\gamma)).$$

Since γ is a shortest curve from $\mathbf{0}$ to $q = f(q)$, $\|\gamma'(s)\| = \|f_* \gamma'(s)\|$ and $\gamma'(s) \in span\{e_1, \gamma(s)\}$ for almost all $s \in (0, \ell(\gamma)]$.

$$(\gamma^N)'(s) = \gamma'(s)^N = \lambda(s)\gamma^N(s), \text{ for } s \in (0, \ell(\gamma)], a.e.$$

$$\frac{d}{ds}(\gamma^N(s)(\gamma^N(s)\cdot\gamma^N(s))^{-\frac{1}{2}}) = 0, \text{ for } s \in (0, \ell(\gamma)], a.e.$$

By absolute continuity and $\gamma(\ell(\gamma))^N = q^N = \left\|q^N\right\| e_2$, one obtains that $\gamma \subset span\{e_1, e_2\}$. This reduces the proof to the $\mathbf{R}^2$ case.

Subcase 2.1. $\|q - e_2\| = 1$, that is $q \in \partial O^c_{\mathbf{0}}(e_1)$. Define

$$g : \{u \in \mathbf{R}^2 : \|u - e_2\| \geq 1\} \to \{u \in \mathbf{R}^2 : \|u - e_2\| = 1\} \text{ by } g(u) = e_2 + \frac{u - e_2}{\|u - e_2\|}.$$

Then, g is a distance decreasing map, $\|g(u) - g(z)\| \leq \|u - z\|$, and equality holds if and only if $\|u - e_2\| = \|z - e_2\| = 1$. Hence, $\ell(\gamma) \geq \ell(g(\gamma))$, and consequently the shortest curve γ must lie on the circle $\|u - e_2\| = 1$ between p and q, by a proof similar to above with f.

Subcase 2.2. $\|q - e_2\| > 1$, that is $q \in intO^c_{\mathbf{0}}(e_1)$. Any component of $\gamma \cap intO^c_{\mathbf{0}}(e_1)$ is a line segment. Let η be the component containing q. By the case assumption and the Case 1, $\bar{\eta} \cap E_1^+ = \emptyset$. There exists unique q' in $\bar{\eta} \cap \partial O^c_{\mathbf{0}}(e_1)$ with $\|q' - e_2\| = 1$. By Case 2.1, γ is a union of a line segment and a circular arc. If γ were not C^1 at $q' = \gamma(t_0)$, then for sufficiently small $\varepsilon > 0$, the line segments between $\gamma(t_0 - \varepsilon)$ and $\gamma(t_0 + \varepsilon)$ lie in $O^c_p(v)$ and have length $< 2\varepsilon$, by the first variation. Hence, γ is C^1, satisfies (a.i-iii) in $\mathbf{R}^2 = span\{e_1, e_2\}$ and consequently in $\mathbf{R}^n$.

Case 3. $\gamma \cap E_1^- \neq \emptyset$. **Subcase 3.1.** $q \in E_1^-$. $q \in intO^c_{\mathbf{0}}(e_1)$. Let η be the line segment component of $\gamma \cap intO^c_{\mathbf{0}}(e_1)$ ending at q. Obviously, $\eta \nsubseteq E_1^-$. Choose $q'' \in (\eta - \{q\} - \partial O^c_{\mathbf{0}}(e_1))$. The part of γ between p to q'' must be shortest also. By Case 2, γ must follow a circular arc to q' then a line segment to q'', which must be along η. This proves (a.i-iii). By rotating γ around E_1, one obtains infinitely many shortest curves γ_α satisfying (a.i-iii).

Subcase 3.2. Suppose there exists $q''' \in \gamma \cap E_1^-$ and $q''' \neq q$. Following any $\gamma_\alpha \neq \gamma$ from p to q''', and γ from q''' to q, would create a shortest curve with a corner within $intO^c_{\mathbf{0}}(e_1)$, an open subset of $\mathbf{R}^n$. Hence, Subcase 3.2 can not occur.

(b) In Case 2, q' and w are unique and γ is unique up to parametrization. □

Proposition 2. *Let $\gamma : I = [0, L] \to \mathbf{R}^n$ be with $\kappa\gamma \leq 1$ and $\|\gamma'\| \equiv 1$. Then,*

(a) $\gamma(s) \in O^c_{\gamma(a)}(\gamma'(a))$, $\forall a, s \in I$ with $|s - a| \leq \pi$.

Furthermore, (rigidity)

($\gamma(s_0) \in \partial O^c_{\gamma(a)}(\gamma'(a))$ for some $a, s_0 \in I$ with $0 < |s_0 - a| \leq \pi$) $\Leftrightarrow$
γ is a circular arc of radius 1 in $\partial O^c_{\gamma(a)}(\gamma'(a))$ between $\gamma(a)$ and $\gamma(s_0)$.

(b) If $\|\gamma(0)\| = \|\gamma(L)\| = 1$ and $\|\gamma(a)\| > 1$ for some $a \in [0, L]$, then $L > \pi$.

(c) If $\gamma''(a)$ exists and $\|\gamma''(a)\| = 1$, for some $a \in [0, L)$, then
$\forall R > 1, \exists\varepsilon > 0$ such that $\gamma((a, a + \varepsilon)) \subset B(\gamma(a) + R\gamma''(a), R)$.

See *Figure 7* for *Proposition 2*.

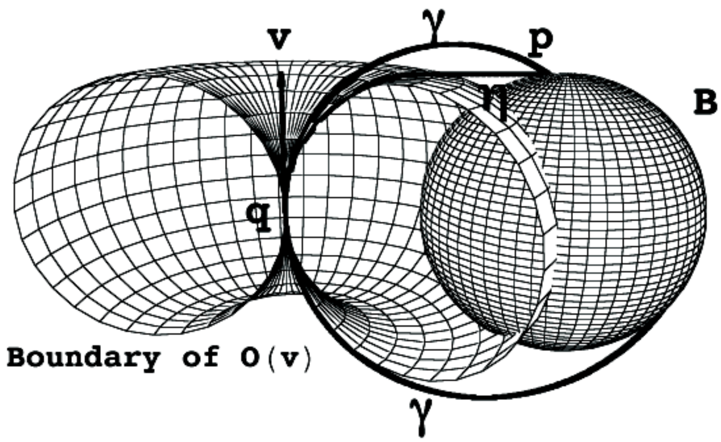

Figure 7. Proposition 2b, $q = \gamma(m)$

Proof. The proof follows the following order: first (a) for $0 \le |s-a| \le \frac{\pi}{2}$ without the rigidity, (b) is next, and then all of (a) for $|s-a| \le \pi$ with the rigidity. (c) is independent.

(a:$\frac{\pi}{2}$) By using an isometry of $\mathbf{R}^n$, reparametrization and symmetry, it suffices to prove this for $a = 0$, $\gamma(0) = \mathbf{0}$, $\gamma'(0) = e_1$ and for $0 \le s \le L' = \min(\frac{\pi}{2}, L)$.

$\gamma'(s) = c_1(s)e_1 + c_2(s)w(s)$ where $\|w(s)\| = 1$ and $w(s) \cdot e_1 = 0$, for $s \in [0, L']$. Then, by *Lemma 1*, $\measuredangle(\gamma'(0), \gamma'(s)) \le s$ and $c_1(s) \ge \cos s$. $|c_2(s)| \le \sin s$, since $c_1^2 + c_2^2 = 1$. For any $u \in \mathbf{R}^n$, define $u^N = u - (u \cdot e_1)e_1$.

$$\gamma(s) \cdot e_1 = \int_0^s \gamma'(t) \cdot e_1 dt \ge \sin s$$
$$\left\|\gamma(s)^N\right\| \le \int_0^s |c_2(t)|\, dt \le 1 - \cos s$$

For any unit vector u normal to e_1,

$\|u - \gamma(s)\|^2 = (\gamma(s) \cdot e_1)^2 + \left\|u - \gamma(s)^N\right\|^2 \ge \sin^2 s + \left\|\left(u - \left\|\gamma(s)^N\right\| u\right)\right\|^2 \ge 1.$ Hence, $\gamma(s) \in O^c_{\mathbf{0}}(e_1)$ for $0 \le s \le L'$.

(b) Choose $m \in (0, L)$ such that $\|\gamma(m)\| \ge \|\gamma(s)\|, \forall s \in [0, L]$. $\|\gamma(m)\| > 1$. We will prove that $m > \frac{\pi}{2}$. We will use $p = \gamma(0)$, $q = \gamma(m)$, $v = -\gamma'(m)$, O^c for $O^c_q(v)$, M for the line through q parallel to v, H for hyperplane through q normal to v, and B for $B(\mathbf{0}, 1)$.

Suppose that $m \le \frac{\pi}{2}$. Then, $\gamma([0, m]) \subset O^c$ by part (a:$\frac{\pi}{2}$). Since $p \in \partial B \cap O^c$, consider a shortest curve η in O^c from q to p and in the direction of v at q, described as in *Proposition 1*. $\ell(\eta) \le \ell(\gamma([0, m])) = m \le \frac{\pi}{2}$. Since q is a furthest point of γ from $\mathbf{0}$, $v \cdot q = \gamma'(m) \cdot \gamma(m) = 0$, and hence, $\mathbf{0} \in H$. $M \cap \overline{B} = \emptyset$, since $\|q\| > 1$. By *Proposition 1* and $p \notin M$, η lies in a unique 2-plane X through q and p, parallel to v, and η is a $C^{1,1}$ curve following a circular arc of length θ_0 of radius 1 and a line segment to p. Each of O^c, X, M and B is symmetric with respect to H, and X intersects ∂B and ∂O^c along circles.

Consider an isometry $f : X \to \mathbf{R}^2$ so that $f(H \cap X)$ and $f(M)$ are x and y axes, respectively, and $f(\eta)$ satisfies $x > 0$. Every point $(x, y) \in f\left(O^c \cap \overline{B} \cap X\right)$ satisfies

$(x-1)^2+y^2 \geq 1$ and $(x-a)^2+y^2 \leq r^2$. One has $0 \leq r < a$ and $r \leq 1$, since $M \cap \overline{B} = \emptyset$ and $\mathbf{0}$ is not necessarily on X. If $r < a \leq 1$, then the above inequalities have no solution in $\mathbf{R}^2$ (by $1 \leq \|(x,y)-(1,0)\| \leq \|(x,y)-(a,0)\| + (1-a) \leq r+1-a < 1$). Consequently, $r \leq 1 < a$ must hold, since $f(p) \in f(O^c \cap \overline{B} \cap X)$. For $f(p) = (x_0, y_0)$,

$$0 \leq 1 - r^2 \leq (x_0-1)^2 + y_0^2 - (x_0-a)^2 - y_0^2 = (a-1)(2x_0 - a - 1)$$
$$x_0 \geq \frac{a+1}{2} > 1, \text{ since } a > 1.$$

The circular part of $f(\eta)$ has length $\theta_0 \leq m \leq \frac{\pi}{2}$ and it goes along the circle $(x-1)^2+y^2 \geq 1$ starting at $(0,0)$. It has to follow a tangent line segment of length at least $\tan(\frac{\pi}{2} - \theta_0) \geq \frac{\pi}{2} - \theta_0$ to reach $x = 1$ line before $f(p)$. This contradicts with $\ell(f(\eta)) \leq m \leq \frac{\pi}{2}$, since $x_0 > 1$. This proves that one has to have $m > \frac{\pi}{2}$, and $L - m > \frac{\pi}{2}$ similarly.

(a:π) It suffices to prove this for $a = 0$, $\gamma(0) = \mathbf{0}$, $\gamma'(0) = e_1$ and $0 \leq s \leq \min(\pi, L)$. Suppose that $\gamma(b) \in O_{\mathbf{0}}(e_1)$ for some $b \in (0,\pi] \cap I$. Then $\gamma(b) \in B(w,1)$ for some unit vector w normal to e_1. There is a unique $c \in [0,b)$ such that $\gamma((c,b]) \subset B(w,1)$ and $\gamma(c) \in \partial B(w,1)$. One must have $\gamma([0,c]) \subset \overline{B(w,1)}$ by part (b), since $\mathbf{0}$ and $\gamma(c)$ are in $\partial B(w,1)$ and $c < \pi$. $\gamma'(c)$ is tangent to $\partial B(w,1)$, since $\|\gamma(t) - w\|$ has a local maximum at $t = c$ when $c \neq 0$, and $c = 0$ case is obvious. By part (a:$\frac{\pi}{2}$), $\gamma(t)$ must stay out of $O_{\gamma(c)}(\gamma'(c)) \supset B(w,1)$ for $t \in [c, c+\frac{\pi}{2}] \cap I$, which contradicts $\gamma((c,b]) \subset B(w,1)$. Hence, $\gamma([0,\min(\pi,L)]) \cap O_{\mathbf{0}}(e_1) = \emptyset$.

Assume that $\exists s_0 \in I$ with $\gamma(s_0) \in \partial O^c_{\mathbf{0}}(e_1)$ and $0 < s_0 \leq \pi$. Then, both $\gamma(s_0)$ and $\mathbf{0}$ are on $\partial B(w_0,1)$ where $w_0 = \gamma(s_0)^N \left\|\gamma(s_0)^N\right\|^{-1}$ is a unit vector normal to e_1. By part (b) and the previous paragraph, $\gamma([0,s_0]) \subset \overline{B(w_0,1)} \cap O^c_{\mathbf{0}}(e_1)$ which is a circle, and $\gamma(s) = (\sin s)\, e_1 + (1-\cos s) w_0$. The converse is obvious.

(c) Let $q = \gamma(a) + R\gamma''(a)$, and define $f(s) = \frac{1}{2}\|\gamma(s) - q\|^2$.

$f'(s) = \gamma'(s) \cdot (\gamma(s) - q)$ which is lipschitz, and $f'(a) = \gamma'(a) \cdot (-R\gamma''(a)) = 0$, by $\|\gamma'\| \equiv 1$.

$f''(s) = \gamma''(s) \cdot (\gamma(s) - q) + \gamma'(s) \cdot \gamma'(s)$ a.e., and $f''(a) = \gamma''(a) \cdot (-R\gamma''(a)) + 1 < 0$.

Hence, $\underset{s \to a^+}{lim} \frac{1}{s-a}(f'(s) - f'(a)) < 0$. There exists $\varepsilon > 0$ such that $f'(s) < 0$, $\forall s \in (a, a+\varepsilon)$. Consequently, $f(s) < f(a)$, $\forall s \in (a, a+\varepsilon)$. □

Example 1. *The constant π in Proposition 2 is sharp. For (b), consider the part of the circle C_ε, $(x-\varepsilon)^2 + y^2 = 1$ in $\mathbf{R}^2$ outside the disc $x^2 + y^2 \leq 1$, for small ε. For (a), consider the LC curve in $\mathbf{R}^2$, which follows C_ε counterclockwise until it reaches $(\varepsilon, 1)$ and then the line segment $[0,\varepsilon] \times \{1\}$ backwards to $(0,1)$.*

Lemma 2. *For all $C^{1,1}$ curves $\gamma : I \to \mathbf{R}^n$, analytic and geometric focal distances are the same: $F_g(\gamma) = F_k(\gamma) := (\sup_I \kappa\gamma(s))^{-1}$. If $\|\gamma'(s)\| = 1$ on I, then $F_g(\gamma) = F_k(\gamma) = (ess \sup \|\gamma''\|)^{-1}$.*

Proof. Reparametrize γ to assume that $\|\gamma'(s)\| = 1$. $\frac{1}{F_k(\gamma)} \geq \kappa\gamma$. By *Proposition 2(a:$\frac{\pi}{2}$)* and rescaling, $\forall p \in \gamma$, γ locally avoids $O_p(F_k(\gamma);\gamma)$ near p and $F_k(\gamma) \leq F_g(p) = \inf\{r > 0 : p \in \overline{O_p(r;\gamma) \cap \gamma}\}$. Hence, $F_k(\gamma) \leq F_g(\gamma) = \inf_{p \in \gamma} F_g(p)$.

Suppose that $F_k(\gamma) < F_g(\gamma)$, i.e. $\sup \kappa\gamma > \frac{1}{F_g(\gamma)}$. Let

$$A = \left\{ s \in I : \kappa\gamma(s) > \frac{1}{F_g(\gamma)} \right\} \text{ and } B = \{s \in I : \gamma''(s) \text{ exists}\}.$$

$A \neq \emptyset$ and the Lebesgue measure $\mu(I - B) = 0$, by Rademacher's Theorem.

Case 1. $A \cap B \neq \emptyset$. There exists $s_0 \in A \cap B$ such that $c := \|\gamma''(s_0)\| = \kappa\gamma(s_0) > \frac{1}{F_g(\gamma)}$. Choose r such that $\frac{1}{c} < r < F_g(\gamma)$. Let $\eta(s) = c\gamma(\frac{s}{c})$, so that $\|\eta'(s)\| = 1, \forall s$, and $\|\eta''(cs_0)\| = 1$. By *Proposition 2c*, $\eta((cs_0, cs_0 + c\varepsilon)) \subset B(\eta(cs_0) + cr\eta''(cs_0), cr)$ for some $\varepsilon > 0$, since $cr > 1$. Hence, $\gamma((s_0, s_0 + \varepsilon)) \subset B(\gamma(s_0) + r\frac{\gamma''(s_0)}{\|\gamma''(s_0)\|}, r) \subset O_{\gamma(s_0)}(r; \gamma)$. However this contradicts $r < F_g(\gamma)$ by *Definition 5b* of F_g.

Case 2. $A \cap B = \emptyset$. Since γ is $C^{1,1}$, γ' is absolutely continuous, $\gamma''(s)$ exists almost everywhere by Rademacher's Theorem and $\|\gamma''(s)\| = \kappa\gamma(s) \leq \frac{1}{F_g(\gamma)}$ a.e. By Lemma 1, $\frac{1}{F_k(\gamma)} = \sup_I \kappa\gamma(s) \leq \frac{1}{F_g(\gamma)}$ which contradicts $F_k(\gamma) < F_g(\gamma)$.

Neither of the cases is possible, hence one must have $F_k(\gamma) = F_g(\gamma)$. By Lemma 1, $F_k(\gamma) = (ess \sup \|\gamma''\|)^{-1}$. □

Definition 9. *Let $p, q \in \mathbf{R}^n$, $v \in UT\mathbf{R}^n_p$, and $w \in UT\mathbf{R}^n_q$ be given. Define $\mathcal{C}(p, q; v, w)$ to be the set of all $C^{1,1}$ curves $\gamma : [0, L] \to \mathbf{R}^n$ with $\gamma(0) = p$, $\gamma'(0) = v$, $\gamma(L) = q$, $\gamma'(L) = w$, $\|\gamma'\| \equiv 1$, and $\kappa\gamma \leq 1$, where $L = \ell(\gamma)$ is not fixed on $\mathcal{C}$.*

Proposition 3. *There exists a shortest curve in $\mathcal{C}(p, q; v, w)$.*

Proof. Obviously, $\mathcal{C}(p, q; v, w) \neq \emptyset$. Any sequence of curves $\{\gamma_m\}_{m=1}^{\infty}$, with $\ell(\gamma_m) \to \inf\{\ell(\gamma) : \gamma \in \mathcal{C}\}$ has uniformly bounded lengths and all starting at p. Extend all γ_m to a common compact interval by following the lines $q + (s - \ell(\gamma_m))w$ after q. By *Lemma 1*, $\forall\gamma \in \mathcal{C}$, $\|\gamma'(s) - \gamma'(t)\| \leq |s - t|$, and thus, $\mathcal{C}$ is C^1-equicontinuous. $\mathcal{C}$ is C^1-bounded by $\|\gamma'\| \equiv 1$. C^0-equicontinuity and boundedness are obvious. By Arzela-Ascoli Theorem, there exists a subsequence of $\{\gamma_m\}_{m=1}^{\infty}$ uniformly converging to γ_0 in C^1 sense: $(\gamma_m(s), \gamma'_m(s)) \to (\gamma_0(s), \gamma'_0(s))$. $\gamma_0 \in \mathcal{C}$, since all conditions of $\mathcal{C}$ are preserved under this convergence and $\ell(\gamma_m) \to \ell(\gamma_0)$. □

Proposition 4. *Let $\gamma : I = [0, L] \to \mathbf{R}^n$ be a shortest curve in $\mathcal{C}(p, q; v, w)$. Then, $\forall s \in I$, ($\kappa\gamma(s) = 0$ or 1). $\kappa\gamma^{-1}(1)$ is a closed subset of I, and $\kappa\gamma^{-1}(0)$ is countable union of disjoint line segments.*

Proof. By the upper semi-continuity of $\kappa\gamma$ (*Remark 3d*), $\forall\lambda \leq 1$, $\kappa\gamma^{-1}([\lambda, 1])$ is a closed subset of I and $J(\lambda) = \kappa\gamma^{-1}([0, \lambda))$ is countable union of relatively open disjoint intervals in I. Choose any $\lambda < 1$, and $a < b$ in a given component J' of $J(\lambda)$.

Suppose that $\gamma'(a) \neq \gamma'(b)$. Choose any smooth bump function $h : \mathbf{R} \to [0, 1]$ such that $supp(h) \subset [-1, 1]$, $h(0) = 1$, and $\int_{-1}^{1} h(s)ds = 1$. Let h_n be defined by $h_n(\frac{a+b}{2}) = 1$ and $h'_n(s) = n[h(n(s - a)) - h(n(s - b))]$. Then,

$$\lim_{n\to\infty} \int_I h'_n(s)\gamma'(s)ds = \gamma'(b) - \gamma'(a) \neq 0.$$

Fix a sufficiently large n such that $supp(h_n) \subset J'$ and $-\int_{J'} h'_n(s)\gamma'(s)ds := V \neq 0$. Let $\gamma_\varepsilon(s) = \gamma(s) + \varepsilon V h_n(s)$ be a variation of γ. By the First Variation formula, [CE, p6],

$$\frac{d}{d\varepsilon}\ell(\gamma_\varepsilon)|_{\varepsilon=0} = \int_I [V h_n(s)]'\gamma'(s)ds = -\|V\|^2 < 0$$

Hence, for sufficiently small $\varepsilon > 0$, γ_ε is strictly shorter that γ.

For all $s < t$ and $0 < \varepsilon < \frac{1}{2}(\sup |h_n'(u)| \, \|V\|)^{-1}$:

$$dil^d \gamma_\varepsilon'(s,t) = \frac{\|\gamma_\varepsilon'(t) - \gamma_\varepsilon'(s)\|}{\ell_{st}(\gamma_\varepsilon)} \leq \frac{\|\gamma'(t) - \gamma'(s)\| + \varepsilon \|V\| \, |h_n'(t) - h_n'(s)|}{(t-s)(1 - \varepsilon \sup |h_n'(u)| \, \|V\|)}$$
$$\leq (1 + \varepsilon C_1) \frac{\|\gamma'(t) - \gamma'(s)\|}{t-s} + \varepsilon C_2$$

where $C_i = C_i(\|V\|, \sup |h_n'|, \sup |h_n''|)$ for $i = 1, 2$.

By *Remark 3a*, and since $\kappa\gamma \leq \lambda < 1$ on J', for sufficiently small ε, $\kappa\gamma_\varepsilon \leq \frac{1+\lambda}{2} < 1$, and $\gamma_\varepsilon \in \mathcal{C}$. This contradicts the minimality of γ. Consequently, $\gamma'(a) = \gamma'(b)$ and γ' is constant on J'. $\forall \lambda < 1$, $\gamma(J(\lambda))$ is a countable union of disjoint line segments, to conclude that $\gamma(J(1))$ is a countable union of disjoint line segments, and $\kappa\gamma(J(1)) \equiv 0$. $\square$

3.1. **Proof of Theorem 1.** See *Figures 8 and 9* in conjunction with this proof.

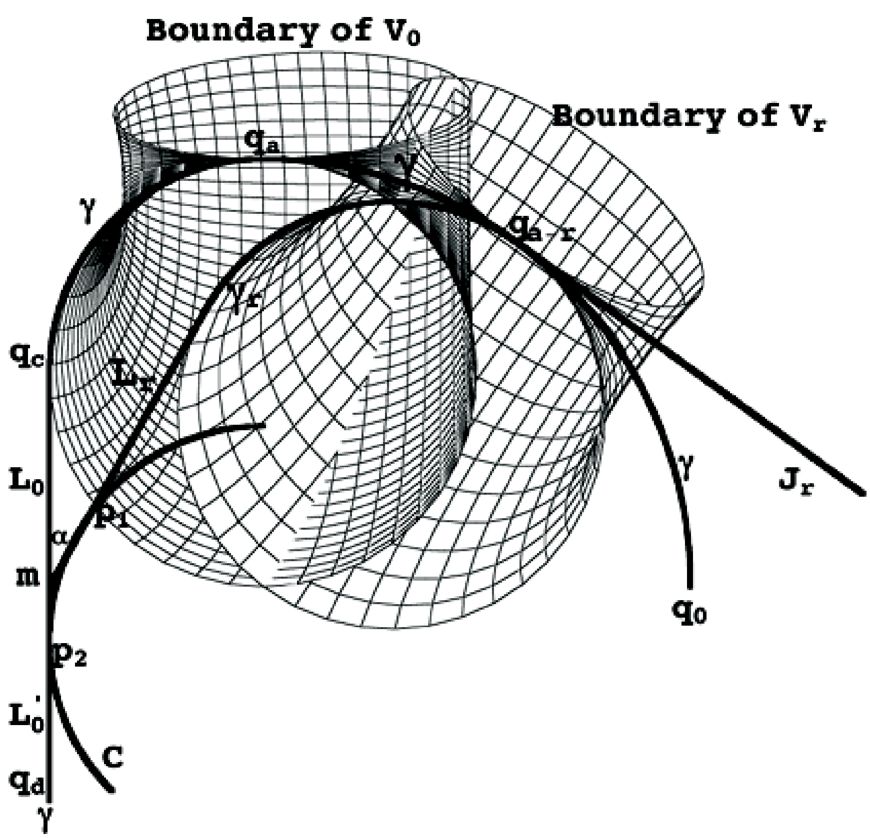

Figure 8. Theorem 1a, for the proof of the impossibility of the case $a > 0$ and $c - a < \pi$, where $q_t = \gamma(t)$

Proof. (a) By *Proposition 4*, there exist maximally chosen c and d such that $s_0 \in [c, d] \subset [0, L]$ and $\gamma([c, d])$ is a line segment L_0 which is a $CLC-$curve. There exist maximally chosen a and b such that $0 \leq a \leq c < d \leq b \leq L$ and $\gamma([a, b])$ still is a

CLC-curve. We want to show that either $a = 0$ or $c - a \geq \pi$. Suppose that $a > 0$ and $c - a < \pi$. Below, we will prove that $\exists \delta > 0$ such that $\gamma([a - \delta, b])$ still is a CLC-curve, contradicting the maximality of $[a, b]$.

Define $J_r = \{\gamma(a - r) - \lambda\gamma'(a - r) : \lambda > 0\}$ and $V_r = O^c_{\gamma(a-r)}(\gamma'(a - r))$, for $r \in [0, a]$.

(i) Let $\varepsilon = \frac{1}{2}\min(a, d - c, \pi - (c - a), 1)$, $c_1 = c + \varepsilon$, and $m = \gamma(c_1)$. $m \in intV_0$, since $\gamma([a, c])$ is an arc of a circle of radius 1 and $\gamma([c, c_1])$ is a line segment, $a \leq c < c_1$, and *Proposition 2a*.

(ii) One obtains that $\forall r \in [0, \varepsilon)$, $\gamma([a - r, c_1]) \subset V_r$ by *Proposition 2a* and $c_1 - (a - r) < c - a + 2\varepsilon \leq \pi$. By *Proposition 2a rigidity,* $m \in intV_r$, since $\gamma([c, c_1]) = L_0$ is a line segment.

For each fixed $r \in [0, \varepsilon)$, define γ_r to be the shortest curve parametrized by arclength from $\gamma(a - r)$ to m within V_r **without curvature restrictions,** by using *Proposition 1*. γ_r follows a circular arc of radius 1 starting from $\gamma(a - r)$ along ∂V_r, then a line segment L_r of positive length until m. In *Proposition 1*, γ'_r at m is not controlled.

Claim 1. $\gamma'_r(m) = \gamma'(m)$ for sufficiently small $r > 0$.

$\exists \delta_1 > 0$ such that $\forall r \in [0, \delta_1)$, $\ell(L_r) \geq \frac{\varepsilon}{2}$, $d(m, J_r) > 0$, since J_r and ∂V_r change continuously in r, $\ell(L_0) = \varepsilon$ and $d(m, J_0) \geq \varepsilon$ (by $0 \leq c - a < \pi$). For $r \in [0, \delta_1)$, γ_r is uniquely defined and depends on r continuously. $\lim_{r\to 0^+} \measuredangle_m(L_0, L_r) = 0$, otherwise one can construct a shortest curve other than γ from $\gamma(a)$ to m in V_0 contradicting *Proposition 1b*. $\exists \delta_2 > 0$ such that $\forall r \in [0, \delta_2)$, $\measuredangle_m(L_0, L_r) \leq 2\tan^{-1}\frac{\varepsilon}{2}$.

Let $\delta = \min(\varepsilon, \delta_1, \delta_2)$. $\forall r \in [0, \delta)$, define a curve $\tilde{\gamma}_r$ which follows γ from p to $\gamma(a - r)$, then γ_r from $\gamma(a - r)$ to m, and γ from m to q. $\tilde{\gamma}_r$ is C^1 at $\gamma(a - r)$, squeezed by $O_{\gamma(a-r)}(\gamma'(a - r))$. Recall that $\varepsilon \leq \frac{d-c}{2}$ and $L_0 = \gamma([c, c_1])$. Define the line segment $L'_0 := \gamma([c_1, c_1 + \frac{\varepsilon}{2}])$.

Suppose that $\tilde{\gamma}_r$ is not C^1 at m for some $r \in (0, \delta)$, that is $\measuredangle_m(L_0, L_r) = \pi - \measuredangle_m(L'_0, L_r) := \alpha > 0$. Fix such an r. $\tilde{\gamma}_r \cap B(m, \frac{\varepsilon}{2})$ is a union of two segments of length $\frac{\varepsilon}{2}$, joined at m with an angle of $\pi - \alpha$, in $L'_0 \cup L_r$. There exists a unique circle C of radius 1 in the same 2-plane as $L'_0 \cup L_r$, tangent to L_r at p_1 and tangent to L'_0 at p_2 where $\|p_i - m\| \leq \frac{\varepsilon}{2}$, since $\alpha \leq 2\tan^{-1}\frac{\varepsilon}{2}$. Let $\tilde{\gamma}$ be the C^1 curve obtained from $\tilde{\gamma}_r$ by replacing the part of $L'_0 \cup L_r$ between p_1 and p_2 by the shorter arc of C between p_1 and p_2.

$$\ell(\tilde{\gamma}) < \ell(\tilde{\gamma}_r) \leq \ell(\gamma) \text{ and } \kappa\tilde{\gamma} \leq 1$$

This contradicts the minimality of γ in $\mathcal{C}$. Hence, $\forall r \in [0, \delta)$, $\tilde{\gamma}_r$ is C^1 at m, and $\measuredangle_m(L_0, L_r) = 0$. This proves Claim 1.

For each given $r \in (0, \delta)$:

1. $\tilde{\gamma}_r \in C^1$ and $\kappa\tilde{\gamma}_r \leq 1$, hence $\tilde{\gamma}_r \in \mathcal{C}$ and $\ell(\tilde{\gamma}_r) \geq \ell(\gamma)$.
2. γ and $\tilde{\gamma}_r$ follow the same path before $\gamma(a - r)$ as well as after m.
3. $\gamma([a - r, c_1]) \subset V_r$, γ_r is the unique shortest curve from $\gamma(a - r)$ to m in V_r, and hence $\ell(\gamma([a - r, c_1])) \geq \ell(\gamma_r([a - r, c_1]))$.

Consequently, $\ell(\tilde{\gamma}_r) = \ell(\gamma)$, γ and $\tilde{\gamma}_r$ are equal up to parametrization, and $\forall r \in [0, \delta)$, $\gamma|[a - r, b])$ is a CLC-curve. Since this contradicts the maximality of $[a, b]$, we must have either $a = 0$ or $c - a \geq \pi$. By using the symmetry, we must also have $L = b$ or $d - b \geq \pi$.

(b) $R_O(\gamma) \geq 1$ implies that $\kappa\gamma \leq 1$ by *Proposition 2c, Lemma 1* and the definition of R_O. Choose a, b, c, d maximally as in part (a). Let X be $\{x \in \mathbf{R}^n : \exists t \in [c, d]$

such that $x - \gamma(t) \perp \gamma([c,d])$ and $0 < \|x - \gamma(t)\| < 2\}$, the solid cylindrical tube with central axis removed. Then, $X \subset O(1, \gamma)$, and hence $X \cap \gamma = \varnothing$ by $R_O(\gamma) \geq 1$.

Suppose that $c - a > \pi$. Then, $\gamma([a,d])$ is a C^1-concatenation of a line segment with a circular arc of strictly more that π radian angle, contained in a unique 2 plane. For sufficiently small $r > 0$, $\|\gamma(c - \pi - r) - \gamma(c + \sin r)\| = 1 + \cos r < 2$ and $\gamma(c - \pi - r) - \gamma(c + \sin r) \perp \gamma([c,d])$, contradicting $X \cap \gamma([0, \pi - c)) = \varnothing$. Consequently, $c - a \leq \pi$.

Suppose that $a > 0$. Then, $c = a + \pi$, by part (a) and previous paragraph. $\gamma([0,a]) \cap X = \varnothing$ follows similarly. We repeat the proof of part (a) with two modifications at (i) and (ii) only. At (i), choose $\varepsilon = \frac{1}{2}\min(a, d - c, 1)$. At (ii) the inequality $c_1 - a + r \leq \pi$ is not needed, since $\forall r \in [0, \varepsilon)$, $\gamma([a - r, c_1]) \subset V_r$ holds by $R_O(\gamma) \geq 1$. The remainder of the proof reaches to a contradiction with the maximality of a, as in part (a) when $a > 0$. Consequently, the final conclusions are $a = 0$ and $c - a \leq \pi$, and by a similar argument, $b = L$ and $b - d \leq \pi$. □

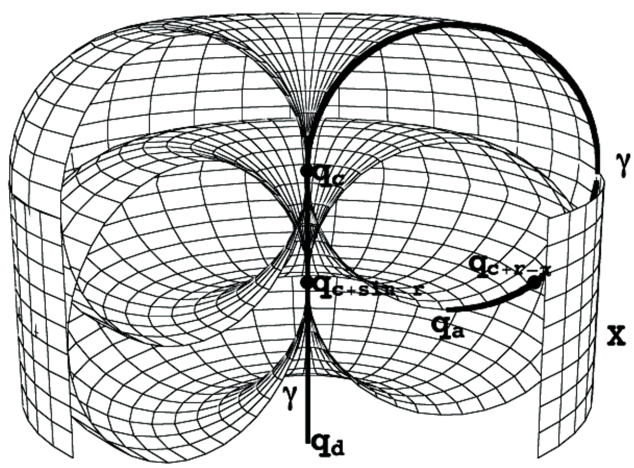

Figure 9. Theorem 1b, for the proof of the impossibility of the case $c - a > \pi$ when $R_0(\gamma) \geq 1$, where $q_t = \gamma(t)$

Example 2. *Consider* CLC-*curves* $\gamma : [a,b] \to \mathbf{R}^n$ *with the circle arcs* C_1, C_2 *and segment piece* L *and each piece has positive length. Each of* $C_i \cup L$ *is contained in a unique 2-plane for* $i = 1, 2$. $p = \gamma(a) \notin L$ *and* $q = \gamma(b) \notin L$.

a. See Figure 10 for a non-planar case.

b. Consider another case that has $p = q$ *and* $v = \gamma'(a) = -\gamma'(b) = -w$. *See Figure 11. Then, all of* γ *is contained in the same 2-plane. There is only one possibility, that is,* C_1 *and* C_2 *are tangent to each other at* p, *each* $\ell(C_i) = \frac{3\pi}{2}$ *and* $\ell(L) = 2$. *The length of the CLC curve* γ *is* $3\pi + 2$. *It is left to the reader to show that any of the pieces* C_1, C_2, L *to have length* 0 *is not possible. However, there is shorter planar CCC−curve,* γ_2 *of length* $\frac{7\pi}{3}$, *as shown in Figure 11.* γ_2 *is the shortest curve for the initial data above when problem is considered in* $\mathbf{R}^2$, *but it is not known to the author that it is the (or a) shortest curve in* $\mathbf{R}^n$ *for* $n \geq 3$. *By Sussmann [S] and the argument above, a shortest curve for this initial data must be CCC or a helicoidal curve when* $n = 3$.

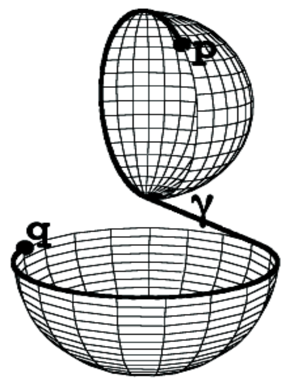

Figure 10. A nonplanar $CLC-$curve

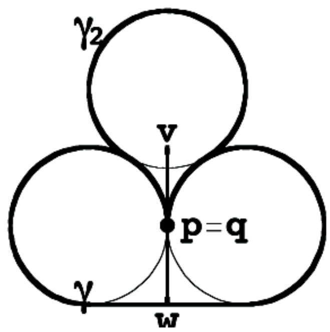

Figure 11. The shortest curve in the plane for the $p = q$ and $v = -w$ initial data is a $CCC-$curve, not a $CLC-$curve.

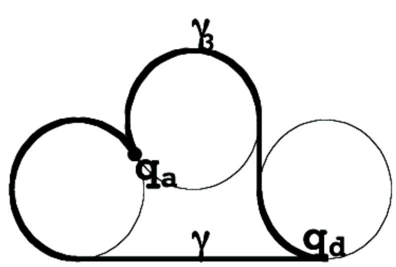

Figure 12. If the line segment is too long in a $U-$bend γ (a circular arc of more than π radians), then the minimizing property may be lost: $\ell(\gamma_3) < \ell(\gamma)$.

Example 3. *a. The $CLC-$curve $\gamma : [a, d] \to \mathbf{R}^n$ in Figure 9 (even though $R_O(\gamma) < 1$) is the shortest curve with $\kappa\gamma \leq 1$ and $c - a \geq \pi$ (U-bend) for the given C^1*

boundary data at $t = a$ and d, as long as $d-c$ is small, by Proposition 1 (a.iii and b). However, as $d-c$ becomes larger, some other shorter curves γ_3 will appear. See Figure 12, drawn in the 2-plane containing γ and γ_3.

b. If $\|p-q\|$ is large, and $p-q, v, w$ are linearly independent, then the shortest curves in $\mathcal{C}(p,q;v,w)$ will be non-coplanar CLC*-curves, by studying the shortest curves in $\mathbf{R}^n - (O_p(v) \cap O_q(w))$ in a similar fashion to Proposition 1. See Figure 10. If $0 < \ell(C_i) < \pi$ (no U-bends) and $0 < \ell(L)$, then the CLC-curves are expected to be the unique minimizers.*

Remark 4. *a. The classification in dimension 3 by Sussmann, Theorem 1 [S], implies the nonexistence of interior CLC sections (proper open subset) of a minimizer γ in $\mathcal{C}(p,q;v,w)$, if $L \neq \varnothing$. In other words, if a minimizer contains a CLC part with $L \neq \varnothing$, then the minimizer itself is CLC with possibly longer circular arcs. We should remark that the nonexistence of LCC or CCL (with $L \neq \varnothing$) minimizers in $\mathbf{R}^3$ are actually proved in the proof of Theorem 1 of Sussmann, even though that is not explicitly mentioned in the statement. Hence, except the trivial cases with a line segment of length 0 in a CCC curve, see Figure 11, there can not exist any examples of interior CLC sections in $\mathbf{R}^3$. In higher dimensions, it is obvious that $C..CLC..C$-curves are not minimizers (if $L \neq \varnothing$), since any section CCL lies in a 3-dimensional subspace, and it is not a minimizer there. However, the minimality of other concatenations, such as helicoidal curves with CLC curves has not been studied in higher dimensions yet.*

b. Sussmann's Theorem 2 [S] states that the helices are local strict minimizers in $\mathbf{R}^3$ if the torsion $|\tau| < 1$ and $\kappa = 1$. Hence, there are shortest curves in $\mathcal{C}(p,q;v,w)$, which do not contain any interior CLC sections. However, [S] contains some remarks about the proof of Theorem 2, but not all details are shown.

4. Relatively Extremal Knots and Links in $\mathbf{R}^n$

In this section, K denotes a union of finitely many disjoint $C^{1,1}$ simple closed curves in $\mathbf{R}^n$ and $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ denotes a one-to-one non-singular parametrization, where $\mathbf{D} = \bigcup_{i=1}^k \mathbf{S}^1_{(i)}$, a union of k copies disjoint circles, unless stated otherwise. When $\|\gamma'\| \equiv 1$ is assumed, $\mathbf{S}^1_{(i)}$ are taken with the appropriate radius and length.

A knot or link class $[\theta]$ is a free C^1 (ambient) isotopy class of embeddings of $\gamma : \mathbf{D} \to \mathbf{R}^n$ with a fixed number of components. Since all of our proofs involve local perturbations of only one component at a time, we will work with $\gamma_{(i)} : \mathbf{S}^1_{(i)} \to \mathbf{R}^n$ and we will omit the lower index $_{(i)}$ to simplify the notation wherever it is possible. We will identify $\mathbf{S}^1 \cong \mathbf{R}/L\mathbf{Z}$, for $L > 0$, and use interval notation to describe connected proper subsets of $\mathbf{R}/L\mathbf{Z}$. In other words, $\gamma_{(i)}(t+L) = \gamma_{(i)}(t)$ and $\gamma'_{(i)}(t+L) = \gamma'_{(i)}(t)$, $\forall t \in \mathbf{R}$ with $\left\|\gamma'_{(i)}\right\| \neq 0$ and $\gamma_{(i)}$ is one-to-one on $[0, L)$.

The following notion of ropelength has been defined and studied by several authors, Litherland-Simon-Durumeric-Rawdon (its reciprocal was called "thickness" in [LSDR]), Gonzales and Maddocks [GM], Cantarella-Kusner-Sullivan [CKS] and others. Cantarella-Kusner-Sullivan [CKS] defined ideal (thickest) knots as "tight" knots.

Definition 10. *For any $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$, one defines the ropelength or extrinsically isoembolic length to be $\ell_e(\gamma) = \ell_e(K) = \frac{\ell(K)}{R_o(K)} = \frac{vol_1(K)}{NIR(K,\mathbf{R}^n)}$.*

Definition 11. *(a) A* $\gamma_0 : \mathbf{D} \to K_0 \subset \mathbf{R}^n$ *is called ideal (or thickest or tight [CKS]) in a knot/link type* $[\gamma_0]$, *if* $\forall \gamma \in [\gamma_0]$ *with* $\gamma : \mathbf{D} \to K$ *one has* $\ell_e(\gamma_0) \leq \ell_e(\gamma)$. $K_0 \subset \mathbf{R}^n$ *is called ideal if any of its parametrizations* γ_0 *is ideal.*

(b) A $\gamma_0 : \mathbf{D} \to K_0 \subset \mathbf{R}^n$ *is called relatively extremal or relatively minimal, if there exists an open set* $\mathcal{U}$ *in* C^1 *topology such that* $\gamma_0 \in \mathcal{U}$ *and* $\forall \gamma \in \mathcal{U} \cap [\gamma_0]$ *with* $\gamma : \mathbf{D} \to K$ *one has* $\ell_e(\gamma_0) \leq \ell_e(\gamma)$. $K_0 \subset \mathbf{R}^n$ *is called relatively extremal /minimal if any of its parametrizations* γ_0 *is relatively extremal/minimal.*

Since both length and thickness are independent of orientations and parametrizations, we have the freedom of choosing the parametrizations to secure isotopies and C^1 convergence. We consider γ_1 and γ_2 to be geometrically equivalent, if there exists an orientation preserving $h : \mathbf{R}^n \to \mathbf{R}^n$, a composition of an isometry and a dilation ($\mathbf{x} \to \lambda \mathbf{x}$, $\lambda \neq 0$) of $\mathbf{R}^n$, such that $h(\gamma_1) = \gamma_2$ up to a reparametrization. On each geometric equivalence class of $C^{1,1}$-closed curves, l_e remains constant.

Lemma 3. *Let* $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$ *be a* C^1 *knot or link.*
(a) If $DCSD(K) > 0$, *then there exists a critical pair* $\{p_0, q_0\}$ *such that* $DCSD(K) = \|p_0 - q_0\|$.
(b) If $\sup \kappa\gamma < \infty$, *i.e.* γ *is* $C^{1,1}$, *then* $DCSD(K) > 0$.

Proof. For every sequence of critical pairs $\{p_m, q_m\}$ with $\|p_m - q_m\| \to DCSD(K)$, one extracts a convergent subsequence (denoted by the same index) such that $p_m \to p_0$ and $q_m \to q_0$, by compactness. Then $0 = (p_m - q_m)\cdot\gamma'(p_m) \to (p_0 - q_0)\cdot\gamma'(p_0) = 0$, as $m \to \infty$, and $\|p_0 - q_0\| = DCSD(K)$. In order $\{p_0, q_0\}$ to be a minimal double critical pair, $p_0 \neq q_0$ is necessary.

For (a), $\|p_m - q_m\| \geq DCSD(K) > 0$ is sufficient to show that $p_0 \neq q_0$.

For (b), dilate and reparametrize γ so that with $\|\gamma'\| = 1$ and $\kappa\gamma \leq 1$. Suppose that $p_m - q_m \to \mathbf{0}$. $\exists m_0$ such that $\forall m \geq m_0$, $\|p_m - q_m\| < 1$ and p_m and q_m belong the same component of K since the distances between different (compact) components of K has a positive lower bound. By *Proposition 2*, $\forall m \geq m_0$, and for $\left|s - \gamma^{-1}(p_m)\right| \leq \pi$, $\gamma(s) \in O^c_{p_m}(\gamma'(p_m))$. But $q_m \in O_{p_m}(\gamma'(p_m))$ since $0 = (p_m - q_m)\cdot\gamma'(p_m)$ and $\|p_m - q_m\| < 1$. Hence the length of γ between p_m and q_m is $\geq \pi$. Same is true between p_0 and q_0. Consequently, $\|p_0 - q_0\| = DCSD(K) > 0$. □

Proposition 5. *Let* $\{\gamma_m\}_{m=1}^{\infty} : \mathbf{D} \to \mathbf{R}^n$ *be a sequence uniformly converging to* γ *in* C^1 *sense, i.e.* $(\gamma_m(s), \gamma_m'(s)) \to (\gamma(s), \gamma'(s))$ *uniformly on* $\mathbf{D}$. *Let* $K_m = \gamma_m(\mathbf{D})$ *for* $m \geq 1$ *and* $K = \gamma(\mathbf{D})$.

(a) ([CKS, Lemma 3] and [L]) If $R_O(K_m) \geq r$ *for sufficiently large* m, *then* $R_O(K) \geq r$. *Consequently,* $\limsup_m R_O(K_m) \leq R_O(K)$.

(b) If $\liminf_m DCSD(K_m) > 0$, *then* $\liminf_m DCSD(K_m) \geq DCSD(K)$.

Proof. (a) The lower semi-continuity of thickness was established earlier by several authors, [CKS, Lemma 3] and [L], and their proofs immediately generalize to $\mathbf{R}^n$. We provide a proof for the sake of completeness and to emphasize the contrast of (a) and (b), which we use repeatedly.

Suppose that $R_O(K) < r$, for a given $r > 0$. By the definition R_O, there exists $a \in \mathbf{D}$, $v \in \mathbf{R}^n$ with $\|v\| = 1$ and $v\cdot\gamma'(a) = 0$ such that $B(\gamma(a) + rv, r) \cap K \neq \emptyset$. One can find $\gamma(b) \in B(\gamma(a) + rv, r - \varepsilon)$ for sufficiently small $\varepsilon > 0$. Choose a sequence $\{v_m\}_{m=1}^{\infty}$ in $\mathbf{R}^n$ such that $\forall m, \|v_m\| = 1$, $v_m \cdot \gamma_m'(a) = 0$, and $v_m \to v$. Then for sufficiently large m,

$$\|(\gamma_m(a) + rv_m) - \gamma_m(b)\| < \|(\gamma(a) + rv) - \gamma(b)\| + \varepsilon < r.$$

Hence, $B(\gamma_m(a)+rv_m, r)\cap\gamma_m \neq \emptyset$ and $R_O(K_m) < r$, for sufficiently large m, which contradicts the hypothesis. Consequently, $R_O(K) \geq r$.

(b) We will use the same indices for subsequences. Let $a = \liminf_m DCSD(K_m)$ and choose a subsequence with $a = \lim_m DCSD(K_m)$ and $DCSD(K_m) > 0, \forall m$. For each m, there exists a minimal double critical pair $\{p_m, q_m\}$ for K_m, $\|p_m - q_m\|$ $= DCSD(K_m)$. Since K is compact and $a > 0$, there exists subsequences $p_m \to p_0 \in K$, $q_m \to q_0 \in K$, where $p_0 \neq q_0$. Since $0 = (p_m - q_m)\cdot\gamma'_m(p_m) \to (p_0 - q_0)\cdot\gamma'(p_0) = 0$, as $m \to \infty$, $\{p_0, q_0\}$ is a double critical pair.

$$DCSD(K) \leq \|p_0 - q_0\| = \lim_m \|p_m - q_m\| = \lim_m DCSD(K_m) = a.$$

□

Definition 12. *For $\gamma : \mathbf{D} \to K \subset \mathbf{R}^n$, (with $\gamma' \neq 0$), define*

(a) $I_c = \{x \in \mathbf{D} : \exists y \in \mathbf{D}$ such that $\|\gamma(x) - \gamma(y)\| = DCSD(K)$ and $(\gamma(x) - \gamma(y)) \cdot \gamma'(x) = (\gamma(x) - \gamma(y)) \cdot \gamma'(y) = 0\}$ and $K_c = \gamma_c = \gamma(I_c)$

(b) $I_z = \{x \in \mathbf{D} : \kappa\gamma(x) = 0\}$ and $K_z = \gamma_z = \gamma(I_z)$

(c) $I_{mx} = \{x \in \mathbf{D} : \kappa\gamma(x) = 1/R_O(K)\}$ and $K_{mx} = \gamma_{mx} = \gamma(I_{mx})$

(d) $I_b = \{x \in \mathbf{D} : 0 < \kappa\gamma(x) < 1/R_O(K)\}$ and $K_b = \gamma_b = \gamma(I_b)$

Remark 5. *K_c and K_{mx} are closed subsets of K if it is $C^{1,1}$. To show this for K_c, one can modify the proof of Lemma 3. See the proof of Proposition 8, for K_{mx}.*

This following result was proved earlier in several articles: [CKS, Theorem 7], [GL], [GMSM], It is an immediate consequence of Arzela-Ascoli theorem, and it does not depend on the (finite) dimension. The existence of normal injectivity radius maximizers in fixed isotopy classes is also true in a more general case of Riemannian manifolds by using Gromov's Compactness Theorem, see [D1].

Proposition 6. *([CKS, Theorem 7], [GL], [GMSM]) For any knot/link class $[\theta]$ in $\mathbf{R}^n$, $\exists\gamma_0 \in [\theta]$ such that*

(a) $\forall\gamma \in [\theta]$, $0 < \ell_e(\gamma_0) \leq \ell_e(\gamma)$, and hence

(b) $\forall\gamma \in [\theta]$, $(\ell(\gamma_0) = \ell(\gamma) \Longrightarrow R_O(\gamma_0) \geq R_O(\gamma))$.

Proposition 7. *Let $\{\gamma_m\}_{m=1}^{\infty} : \mathbf{D} \to \mathbf{R}^n$ be a sequence uniformly converging to γ in C^1 sense, $K = \gamma(\mathbf{D})$ and $K_m = \gamma_m(\mathbf{D})$ such that $\exists C < \infty, \forall m, \sup \kappa\gamma_m \leq C$.*

(a) Let $A \subset \mathbf{D}$ be a given compact set with $\{s \in \mathbf{D} : \gamma_m(s) \neq \gamma(s)\} \subset A, \forall m$. If $A \cap I_c = \emptyset$, then $\exists m_1$ such that $\forall m \geq m_1, DCSD(K_m) \geq DCSD(K)$.

(b) If $F_k(K) < \frac{1}{2}DCSD(K)$ and $F_k(K_m) \geq F_k(K), \forall m$, then $\exists m_1$ such that $\forall m \geq m_1, R_O(K_m) \geq R_O(K)$.

Proof. All subsequences will be denoted by the same index m. The critical pairs will be identified from the common domain $\mathbf{D}$.

(a) Suppose there exists a subsequence γ_m such that $DCSD(K_m) < DCSD(K)$, $\forall m$. For all m, there exists a minimal double critical pair $\{x_m, y_m\}$ in $\mathbf{D}$ for γ_m, $DCSD(K_m) = \|\gamma_m(x_m) - \gamma_m(y_m)\| < DCSD(K)$. There exist subsequences $x_m \to x_0$, $y_m \to y_0$, by compactness of $\mathbf{D}$.

$x_0 \neq y_0$, if they are in different components of $\mathbf{D}$. If both x_0 and $y_0 \in \mathbf{S}^1_{(i)}$, then for sufficiently large m, x_m and $y_m \in \mathbf{S}^1_{(i)}$. As in the proof of *Lemma 3*, by *Proposition 2*, for sufficiently large m, the length of γ_m between $\gamma_m(x_m)$ and $\gamma_m(y_m)$

will be at least $\frac{\pi}{C}$. The same is true for γ between $\gamma(x_0)$ and $\gamma(y_0)$, to conclude that $x_0 \neq y_0$. Since $0 = (\gamma_m(x_m) - \gamma_m(y_m)) \cdot \gamma'(x_m) \to (\gamma(x_0) - \gamma(y_0)) \cdot \gamma'(x_0) = 0$, as $m \to \infty$, $\{x_0, y_0\}$ is a double critical pair for K.

$$\begin{aligned} DCSD(K) &\leq \|\gamma(x_0) - \gamma(y_0)\| = \lim_m \|\gamma_m(x_m) - \gamma_m(y_m)\| \\ &= \lim_m DCSD(K_m) \leq DCSD(K) \end{aligned}$$

Hence, $\{x_0, y_0\}$ is a minimal double critical pair for K and $\{x_0, y_0\} \subset I_c$. Since I_c and A are disjoint compact subsets of $\mathbf{D}$, the subsequences $\{x_m\}_{m=1}^{\infty}$ and $\{y_m\}_{m=1}^{\infty}$ can be taken in $\mathbf{D} - A$. $\forall m$, $\{x_m, y_m\}$ is a double critical pair for K, since $\gamma_m = \gamma$ on $\mathbf{D} - A$.

$$DCSD(K) \leq \|\gamma(x_m) - \gamma(y_m)\| = \|\gamma_m(x_m) - \gamma_m(y_m)\| = DCSD(K_m)$$

which contradicts the initial assumption. Consequently, there does not exist any subsequence γ_m such that $\forall m, DCSD(K_m) < DCSD(K)$, proving (a).

(b) By the hypothesis, Thickness formula and *Proposition 5*:

$$2R_O(K) = 2F_k(K) < DCSD(K) \leq \liminf_m DCSD(K_m).$$

Hence, $\exists m_1$ such that $\forall m \geq m_1, DCSD(K_m) > 2R_O(K)$, and $F_k(K_m) \geq F_k(K)$, to conclude

$$R_O(K_m) = \min\left(F_k(K_m), \frac{1}{2} DCSD(K_m)\right) \geq \min\left(F_k(K), R_O(K)\right) = R_O(K).$$

□

Proposition 8. *(Also see [GM, p4771] for another version for smooth ideal knots.) Let K be a $C^{1,1}$ relatively minimal knot or link for the ropelength ℓ_e.*

(a) If $DCSD(K) = 2R_O(K)$, then $K - (K_c \cup K_{mx})$ is a countable union of open ended line segments, and hence $I_b \subset I_c$.

(b) If $DCSD(K) > 2R_O(K)$, then $K - K_{mx}$ is a countable union of open ended line segments.

Remark 6. *Theorem 2 below shows that $K - K_{mx}$ is actually empty when $DCSD(K) > 2R_O(K)$.*

Proof. By using dilations of $\mathbf{R}^n$, one can assume that $\sup \kappa\gamma = 1$. Let $\mathcal{U}$ be an open set in C^1 topology such that $\gamma \in \mathcal{U}$ and $\ell_e(\gamma) \leq \ell_e(\eta), \forall \eta \in \mathcal{U} \cap [\gamma]$.

(a) As in the proof of *Proposition 4*, for all $\lambda \leq 1$, $\kappa\gamma^{-1}([0, \lambda)) - I_c$ is countable union of relatively open intervals or component circles in $\mathbf{D}$. Choose any $\lambda < 1$ and a closed interval $[a, b]$ contained in a component of $\kappa\gamma^{-1}([0, \lambda)) - I_c$. By repeating the proof of *Proposition 4*, if $\gamma|\,[a, b]$ is not a line segment, then there exists a length decreasing variation $\gamma_\varepsilon(s) = \gamma(s) + \varepsilon V h_n(s)$ supported in $[a, b]$. There exists a sufficiently small $\varepsilon_1 > 0$ such that $\forall \varepsilon, 0 < \varepsilon \leq \varepsilon_1$, one has

1. γ_ε and γ belong to the same knot class and $\gamma_\varepsilon \in \mathcal{U}$.
2. $\ell(\gamma_\varepsilon) < \ell(\gamma)$, (proof of *Proposition 4*)
3. $\kappa\gamma_\varepsilon \leq 1$ and hence $F_k(K_\varepsilon) \geq F_k(K) = 1$, (proof of *Proposition 4*), and
4. $DCSD(K_\varepsilon) \geq DCSD(K)$ (*Proposition 7(a)* and $[a, b] \cap I_c = \emptyset$).

By the *Thickness Formula*, $R_O(K_\varepsilon) \geq R_O(K)$ and $\ell_e(\gamma_\varepsilon) = \frac{\ell(\gamma_\varepsilon)}{R_O(K_\varepsilon)} < \frac{\ell(\gamma)}{R_O(K)} = \ell_e(\gamma)$ which contradicts the hypothesis. Hence, $\gamma|\,[a, b]$ must be a line segment. Consequently, $I_b \subseteq I_c$.

(b) We assume that $DCSD(K) > 2R_O(K) = 2F_k(K) = 2$. The proof is essentially the same as in (a), with the following modifications. $[a, b]$ is taken in any component of $\kappa\gamma^{-1}([0, \lambda))$, thus $[a, b] \cap I_c$ may not be empty. (1-3) above hold. To conclude $R_O(K_\varepsilon) \geq R_O(K)$, one uses Proposition 7(b). In this case, $I_b = \emptyset$ and $K - K_{mx}$ is a countable union of open ended line segments. □

4.1. **Proof of Theorem 2.**

Proof. Proof of (a) is a simpler version of (b), and we will prove (b) for a connected $\gamma : \mathbf{D} = \mathbf{S}^1 \to K \subset \mathbf{R}^n$ first. By using dilations of $\mathbf{R}^n$, one can assume that $\sup \kappa\gamma = 1$ and consequently $F_k(K) = 1$, as well as $\|\gamma'\| \equiv 1$. Let $\mathcal{U}$ be an open set in C^1 topology such that $\gamma \in \mathcal{U}$ and $\forall \eta \in \mathcal{U} \cap [\gamma]$, $\ell_e(\gamma) \leq \ell_e(\eta)$.

(b) By *Proposition 8*, there exist maximally chosen c, d such that $\gamma|(c, d)$ is an open ended line segment, $s_0 \in (c, d)$ and $(c, d) \cap I_c = \emptyset$. If $d \in I_c$, then take $b = d$, and there is nothing to prove at this end. If $d \notin I_c$, proceed as follows. Assume that $\gamma|[s_0, d_0]$ is a CLC-curve (in fact, a line segment followed by circular arc of possibly zero length, that is an $LC-curve$) such that $d \leq d_0 < d + \pi$ and $[s_0, d_0] \cap I_c = \emptyset$. We will show that the same is true for $d_0 + \varepsilon$ for some $\varepsilon > 0$. We point out that a priori $\gamma|[s_0, d_0]$ is not known to be a shortest curve in a certain $\mathcal{C}$, replacing it with a shortest curve may create a knot outside $\mathcal{U}$ or the knot class of γ, and this shortest curve may not have a point of zero curvature.

Let $\{d_m\}_{m=1}^{\infty}$ be a sequence and $\eta > 0$ be such that

1. $d_{m+1} < d_m, \forall m \in \mathbf{N}^+$,
2. $d_m \to d_0$,
3. $0 < d_m - d \leq \eta < \pi$, and
4. $[s_0, d + \eta] \cap I_c = \emptyset$.

Since $\gamma(s_0) \in int\, O^c_{\gamma(d_0)}(-\gamma'(d_0))$ *(Proposition 2a)*, $\gamma(s_0) \in int\, O^c_{\gamma(d_m)}(-\gamma'(d_m))$ for sufficiently large m. Let $f_m(s)$ be the unique shortest curve parametrized by arclength in $\mathcal{C}(\gamma(s_0), \gamma(d_m); \gamma'(s_0), \gamma'(d_m))$ by *Propositions 2 and 3,* such that $f_m(s_0) = \gamma(s_0)$ and $f_m(t_m) = \gamma(d_m)$ for some $t_m \in (s_0, d_m]$. Extend f_m to $[s_0, d+\eta]$ in a C^1 fashion beyond $\gamma(d_m)$ by $\gamma(s - t_m + d_m) = f_m(s)$.

For sufficiently large m, $\kappa f_m \leq 1, \|f'_m\| = 1$ and $f_m(s_0) = \gamma(s_0)$. Hence, the sequence $\{f_m\}_{m=1}^{\infty}$ is C^1-equicontinuous and C^1-bounded. By Arzela-Ascoli Theorem, there exists a convergent subsequence (which we will denote by the same subindices m) $f_m \to f_0$ uniformly in C^1 topology. By the construction above, f_0 follows γ past $\gamma(d_0)$ and $f_0(t_0) = \gamma(d_0)$ for some t_0.

$$t_m - s_0 \leq d_m - s_0$$
$$\limsup_m t_m \leq d_0$$
$$t_0 \leq d_0$$

$$f_0(s_0) = \gamma(s_0) \text{ and } f_0(t_0) = \gamma(d_0)$$
$$f_0 \in C^1 \text{ and } f_0'(t_0) = \gamma'(d_0)$$
$$\kappa f_0 \leq 1$$

$\gamma|[s_0, d_0]$ is a line segment followed by a circular arc of length at most π, and by *Proposition 1*, it is the unique shortest curve satisfying the last 3 conditions. Consequently, $d_0 = t_0$ and $f_0 = \gamma$ on $[s_0, d + \eta]$.

Let γ_m be the C^1 curve obtained from γ by replacing $\gamma|\,[s_0, d_m]$ by $f_m|[s_0, t_m]$. Reparametrize γ_m (not necessarily with respect to arclength) so that $\gamma_m(s) = \gamma(s)$ for $s \notin [s_0, d+\eta]$ and $\gamma_m \to \gamma$ in C^1 sense on $\mathbf{S}^1$, which is possible since $\frac{d_m - s_0}{t_m - s_0} \to 1$. Let $K_m = \gamma_m(\mathbf{S}^1)$. For sufficiently large $m \geq m_1$:

1. γ_m and γ belong to the same knot class and $\gamma_m \in \mathcal{U}$.
2. $\{s : \gamma_m(s) \neq \gamma(s)\} \subset [s_0, d + \eta]$ which is disjoint from I_c.
3. $F_k(K_m) \geq F_k(K) = 1$, by construction of f_m.
4. $DCSD(K_m) \geq DCSD(K)$, by *Proposition 7(a).*
5. $R_O(K_m) \geq R_O(K)$, by *Thickness Formula*, (3) and (4).
6. $\ell_e(K_m) \geq \ell_e(K)$, since K is relatively extremal and (1).
7. $\ell(K_m) \geq \ell(K)$ by (5), (6) and the definition of ℓ_e.
8. $\ell(K_m) \leq \ell(K)$ by construction of f_m and γ_m.
9. $f_m|[s_0, t_m]$ and $\gamma|\,[s_0, d_m]$ have the same minimal length in $\mathcal{C}(\gamma(s_0), \gamma(d_m); \gamma'(s_0), \gamma'(d_m))$.
10. $\gamma|[s_0, d_m]$ is a CLC-curve, by *Theorem 1(a)* and $\kappa\gamma([s_0, d]) = 0$.

We proved that if $\gamma|[s_0, d_0]$ is a LC-curve such that $d \leq d_0 < d + \pi$ and $[s_0, d_0] \cap I_c = \emptyset$, then there exists $\varepsilon = d_{m_1} - d_0 > 0$ such that $\gamma|[s_0, d_0 + \varepsilon]$ is a CLC-curve. In fact, $\gamma|[s_0, d_0 + \varepsilon]$ must be an $LC - curve$ by *Theorem 1(a),* since $d_0 + \varepsilon - d < \pi$. Hence, $\delta_0 := \max\{\delta : \gamma|[s_0, d + \delta]$ is a $LC -$ curve$\}$ satisfies that either $d + \delta_0 \in I_c$ or $\delta_0 \geq \pi$.

In the case of $\delta_0 \geq \pi$, $\gamma(d)$ and $\gamma(d+\pi)$ are an antipodal pair on a circle of radius 1, forming a double critical pair. This shows that $\frac{1}{2}DCSD(K) \leq 1$. Suppose that $\frac{1}{2}DCSD(K) < 1 = F_k(K)$ which implies that $R_O(K) = \frac{1}{2}DCSD(K) < 1$. The existence of the semi-circle $\gamma([d, d+\pi])$ would then contradict *Proposition 8a,* (see *Definition 12).* Hence, $\gamma(d)$ and $\gamma(d+\pi)$ must form a *minimal* double critical pair, which implies that $d \in I_c$.

In all cases, there exists a smallest $b \in [d, d+\pi) \cap I_c$ such that $\gamma([s_0, b])$ is a line segment of positive length followed by a circular arc of length $b - d \in [0, \pi)$. The proof is the same for the opposite direction before c.

(a) Suppose that $\frac{1}{2}DCSD(K) > R_O(K) = F_k(K) = 1$. One proceeds as above in (b) by omitting all conditions about avoiding I_c. Use *Proposition 8(b)*, to obtain the line segment $\gamma|(c, d)$. Even though $DCSD(K_m) \geq DCSD(K)$ may not be valid by *Proposition 7(a)*, $R_O(K_m) \geq R_O(K)$ is valid by *Proposition 7(b).* This shows that $\gamma|[s_0, d+\pi]$ is a $CLC-$curve, even passing through DCSD-points. $\gamma(d)$ and $\gamma(d+\pi)$ are an antipodal pair on a circle of radius 1, forming a double critical pair. This shows that $DCSD(K) \leq \|\gamma(d) - \gamma(d+\pi)\| = 2$ which is contrary to the hypothesis. Consequently, the case of $\frac{1}{2}DCSD(K) > R_O(K) = F_k(K) = 1$ with $\exists s_0 \in \mathbf{S}^1, \kappa\gamma(s_0) < \sup \kappa\gamma = 1$ is vacuous.

The generalizations of this proof to K with several components is straightforward, since (b) is a local result, and in (a) the existence of any s_0 with $\kappa\gamma(s_0) < 1$ leads to a contradiction. □

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