Conformal Transformations and Curvature

Richard G. Ligo[a,b] and Oguz C. Durumeric[b]

[a] *Mathematics Department, Gannon University, Erie, PA 16541, USA*

[b] *Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA*

February 18, 2018

Abstract

In this paper we discuss the relationship between conformal transformations of $\mathbb{R}^n \cup \{\infty\}$ and the curvature of curves. First, for any non-circular closed curve, there exists a length-preserving inversion such that the maximum pointwise absolute curvature can be made arbitrarily large. In contrast, we show that the total absolute curvatures of a family of curves conformally equivalent to a given simple or simple closed curve is uniformly bounded. Furthermore, we show that the total absolute curvature of an inverted regular C^2 simple closed curve as a function of inversion center and radius is removably discontinuous along the curve with exactly a 2π drop, and continuous elsewhere.

1 Introduction

In 1990 Jun O'Hara proposed a knot energy functional that would later come to be known as the Möbius energy [7] [8], shown in Definition 1.1.

Definition 1.1. Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^1 curve. Then the *Möbius energy of* γ is defined as

$$E(\gamma) = \int_0^\ell \int_0^\ell \left[\frac{1}{|\gamma(s) - \gamma(t)|^2} - \frac{1}{D_\gamma(s,t)^2} \right] |\gamma'(s)||\gamma'(t)| \, ds \, dt.$$

$D_\gamma(s,t)$ is called the *intrinsic distance*; specifically, $D_\gamma(s,t)$ is the shortest distance between $\gamma(s)$ and $\gamma(t)$ along γ.

Many results have been built out of O'Hara's definition and the foundational work presented by Freedman, He, and Wang's 1994 paper, *Möbius Energy of Knots and Unknots* [5]. Theorem 1.1 concerns the relationship between the Möbius energy and Möbius transformations, and is arguably the most important of their results. Following their convention, we define Möbius transformations to be the set of transformations of $\mathbb{R}^n \cup \{\infty\}$ generated by translations, rotations, reflections, dilations, and inversions. Additionally, affine similarities of $\mathbb{R}^n \cup \{\infty\}$ are generated by translations, rotations, reflections, and dilations. We will represent the inversion through an n-sphere of radius r centered at c as $\mathrm{I}_{r,c}$. Throughout the paper, all curves are assumed to be simple or simple closed, except in Subsubsection 3.4.2.

Theorem 1.1 (Freedman, He, and Wang)**.** *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a closed curve and let T be a Möbius transformation of $\mathbb{R}^n \cup \{\infty\}$. Then we have two cases:*

(i) If $T(\gamma) \subseteq \mathbb{R}^n$, then $E(T(\gamma)) = E(\gamma)$.

(ii) If $\infty \in T(\gamma)$, then $E(T(\gamma) - \{\infty\}) = E(\gamma) - 4$.

In particular, we are interested in the relationship between Möbius transformations and the curvature of curves—specifically, we have investigated both the pointwise absolute curvature and the total absolute curvature, which we denote by $\kappa_\gamma^{\text{abs}}(t)$ and $\kappa_{\text{tot}}^{\text{abs}}(\gamma)$, respectively. Investigation of said relationship resulted in the discovery of Propositions 2.1 and 3.18.

Recall from the Whitney-Graustein Theorem that the total *signed* curvature of a simple plane curve will always be equal to 2π. Notice that the total *absolute* curvature is more sensitive, as for a simple plane curve it will equal 2π if and only if the curve is convex. In fact, any curve can be made to have an arbitrarily large total absolute curvature via a diffeomorphism.

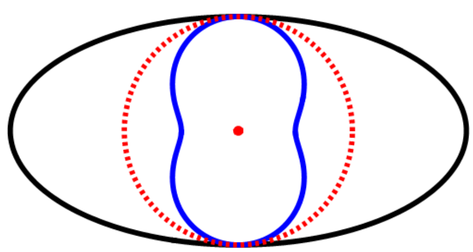

Figure 1: Inversion changes both the pointwise and total absolute curvatures.

Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a simple closed curve, then for some $\ell \in \mathbb{R}^+$ we can define an extension $\hat{\gamma} : \mathbb{R} \to \mathbb{R}^n$ via $\hat{\gamma}(s + \ell) = \hat{\gamma}(s)$ for all s satisfying $\hat{\gamma}|_{[0,\ell]} = \gamma$. As such, we shall use γ and $\hat{\gamma}$ interchangeably where appropriate. In addition, $\hat{\gamma}$ is assumed to be C^2 on $\mathbb{R}$ if γ is said to be C^2 (simple) closed.

Proposition 2.1. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a non-circular, regular, C^2 closed curve. Then for any $K \in \mathbb{R}^+$ there exists a length-preserving Möbius transformation T such that* $\max \kappa^{abs}_{T(\gamma)}(t) > K$.

It is interesting to note that Proposition 2.1 tells us that a family of curves with bounded Möbius energy does not imply bounded pointwise absolute curvature. A weakened converse of this statement is true for the total absolute curvature.

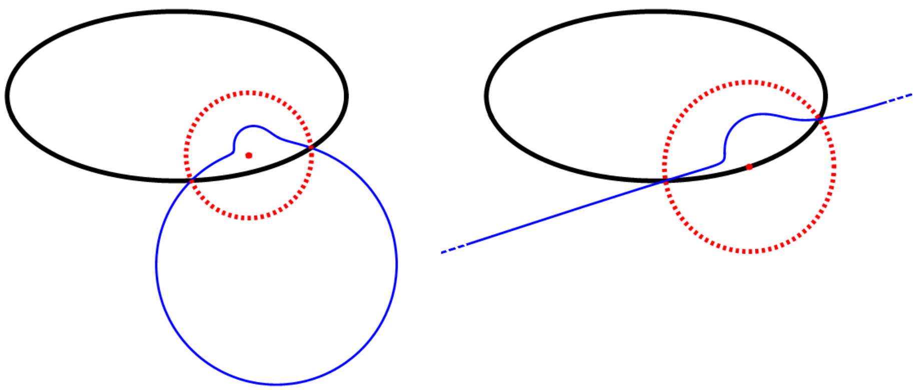

Figure 2: Inversion centers both near and on the curve.

Proposition 3.18. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. Then there exists some $M \in \mathbb{R}^+$ such that $\kappa^{abs}_{tot}(T(\gamma)) \leq M_0$ for all Möbius transformations T.*

Proposition 3.18 can be viewed as a corollary of Theorem 3.19.

Theorem 3.19. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength, and define the map $\hat{\Phi} : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ by*

$$\hat{\Phi}(r, c) = \begin{cases} \kappa^{abs}_{tot}(I_{r,c}(\gamma)) + 2\pi & c \in \gamma([0, \ell]) \\ \kappa^{abs}_{tot}(I_{r,c}(\gamma)) & c \notin \gamma([0, \ell]) \end{cases}.$$

Then $\hat{\Phi}$ is continuous and bounded. In fact,

$$\lim_{|c|\to\infty} \hat{\Phi}(r,c) = \kappa_{tot}^{abs}(\gamma) \quad \text{for all } r \in \mathbb{R}^+.$$

Remark. Note that Theorem 1.1 can be rephrased in a manner that is parallel to Theorem 3.19. Specifically, if $\gamma : [0,\ell] \to \mathbb{R}^n$ is a closed curve, then the map $\hat{\Theta} : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{\Theta}(r,c) = \begin{cases} E(\mathrm{I}_{r,c}(\gamma)) + 4 & c \in \gamma([0,\ell]) \\ E(\mathrm{I}_{r,c}(\gamma)) & c \notin \gamma([0,\ell]) \end{cases}$$

is constant.

Theorem 3.19 can be generalized to C^2 closed curves with finitely many self-intersections—see Subsubsection 3.4.2.

Theorem 3.20. *Let $\gamma : [0,\ell] \to \mathbb{R}^n$ be a regular C^2 closed curve parametrized with respect to arclength, possess finitely many points of self-intersection, and be such that $\gamma^{-1}(c)$ is a finite set for all $c \in \mathbb{R}^n$. Then*

$$\hat{\Psi}(r,c) = \kappa_{tot}^{abs}(I_{r,c}(\gamma)) + 2\pi m(c)$$

is continuous, where $m(c) = card(\gamma^{-1}(c))$; that is, $m(c)$ is the multiplicity of γ at c when $c \in \gamma$ and $m(c) = 0$ if $c \notin \gamma$.

Under inversion, lines do not go to lines in general—this indicates that the usefulness of piecewise linear approximations is limited. A distinctive behavior of polygonal curves in $\mathbb{R}^2$ is described by Proposition 3.24.

Proposition 3.24. *Let $p : [0,\ell] \to \mathbb{R}^2$ be a closed, convex, polygonal curve of n segments $\lambda_1, \ldots, \lambda_n$, where $p(t_i)$ is the initial point of λ_i. Let $r \in \mathbb{R}^+$. If $c \in \mathbb{R}^2$ is in the compact region bounded by the curve, then $\kappa_{tot}^{abs}(I_{r,c}(p)) = 6\pi$.*

Consider an approximation of a circle with a convex polygonal curve, and choose an inversion center in the interior of both. Both curves have total absolute curvature equal to 2π, but the inversion of the unit circle will be another circle (and thus have total absolute curvature equal to 2π), while the inversion of the piecewise approximation will have total absolute curvature 6π as a result of Proposition 3.24. This shows that proving Theorems 3.19 and 3.20 via linear approximations is unlikely, if not impossible.

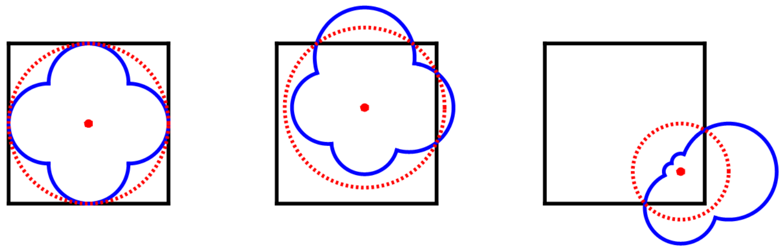

Figure 3: Inversions of squares with total absolute curvature equal to 6π.

2 Pointwise Curvature

We first prove Proposition 2.1.

Proposition 2.1. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a non-circular, regular, C^2 closed curve. Then for any $K \in \mathbb{R}^+$ there exists a length-preserving Möbius transformation T such that $\max \kappa^{abs}_{T(\gamma)}(t) > K$.*

Proof. Since γ is not a circle, there exists some point $s \in [0, \ell]$ such that the osculating circle at $\gamma(s)$ does not coincide with every other point on γ. Set $q = \gamma(s)$, and let $p = \gamma(t)$ be a point on γ not on the osculating circle (or line) at q, which we call O_q. Since $p \notin O_q$, there exists some open n-ball $B^n_\epsilon(p)$ such that $B^n_\epsilon(p) \cap O_q = \emptyset$ and $\frac{1}{\epsilon} > K$. Note that both of these conditions simply require that ϵ is sufficiently small, so we immediately know that such ϵ exists. We claim there exists some $c \notin \gamma$ and $r \in \mathbb{R}^+$ such that

(i) $\mathrm{I}_{r,c}(O_q) = S^1_\delta(x)$, where $\frac{1}{\delta} > K$.

(ii) $\mathrm{length}(\mathrm{I}_{r,c}(\gamma)) = \mathrm{length}(\gamma)$.

To prove the claim we proceed as follows. Choose $c_0 \notin \gamma$ and $r_0 \in \mathbb{R}^+$ so that $S^{n-1}_{r_0}(c_0) \subseteq B^n_\epsilon(p)$. As O_q is disjoint from $B^n_\epsilon(p)$ and smooth Möbius transformations send osculating circles and lines to osculating circles and lines, we know then that $\mathrm{I}_{r_0,c_0}(O_q)$ will be an osculating circle to $\mathrm{I}_{r_0,c_0}(\gamma)$ at $\mathrm{I}_{r_0,c_0}(q)$ inside $S^{n-1}_{r_0}(c_0)$, which we call $S^1_\delta(x)$. Since $S^1_\delta(x) \subseteq B^n_{r_0}(c_0) \subseteq B^n_\epsilon(p)$ and $S^{n-1}_{r_0}(c_0) \subseteq B^n_\epsilon(p)$, we then have that

$$\frac{1}{\delta} > \frac{1}{r_0} > \frac{1}{\epsilon} > K.$$

This follows from comparing the curvature of $S^1_\delta(x)$ to the curvature of circle obtained from the intersection of the 2-plane containing $S^1_\delta(x)$ and $S^{n-1}_{r_0}(c_0)$.

Hence, we have that Condition (i) is satisfied. Note now that

$$\lim_{c \to p} \text{length}(\mathrm{I}_{r_0,c}(\gamma)) = \infty \quad \text{and} \quad \lim_{r \to 0} \text{length}(\mathrm{I}_{r,c_0}(\gamma)) = 0.$$

The first limit comes from the fact that as the center of the inverting circle approaches γ, the image of γ under inversion has points sent further away from p, since r_0 is fixed. In the second limit, c_0 is fixed. Observe then that for all choices of $r \in \mathbb{R}^+$, the images $\mathrm{I}_{r,c_0}(\gamma)$ will all be scale copies of each other. Additionally, since $c_0 \notin \gamma$, we can choose r sufficiently small so that no point of γ is in $B_r^n(c_0)$. Thus, $\mathrm{I}_{r,c_0}(\gamma) \subseteq B_r^n(c_0)$. Combining this with the statement about scale copies gives us the second limit above. Given the c_0, r_0 we have in hand at this point, we have three cases.

Case 1: $\text{length}(\mathrm{I}_{r_0,c_0}(\gamma)) = \text{length}(\gamma)$. Since Conditions (i) and (ii) are satisfied, we need not change our choice of c_0 and r_0.

Case 2: $\text{length}(\mathrm{I}_{r_0,c_0}(\gamma)) > \text{length}(\gamma)$. We can continuously decrease $\text{length}(\mathrm{I}_{r,c_0}(\gamma))$ by continuously decreasing r. (Here the continuity comes from our above statement about scale copies.) Hence, we simply decrease r until Condition (ii) is satisfied. Note that any decrease in r will always preserve $S_r^{n-1}(c_0) \subseteq B_\epsilon^n(p)$, which ensures that Condition (i) remains satisfied.

Case 3: $\text{length}(\mathrm{I}_{r_0,c_0}(\gamma)) < \text{length}(\gamma)$. We can increase $\text{length}(\mathrm{I}_{r_0,c}(\gamma))$ indefinitely by moving c closer to p. Our goal is to choose c sufficiently close to p so $\text{length}(\mathrm{I}_{r,c}(\gamma)) > \text{length}(\gamma)$, which reduces this case to case 2. Note that moving c closer to p always preserves $S_r^{n-1}(c) \subseteq B_\epsilon^n(p)$, which ensures that Condition (i) remains satisfied. As γ is a regular curve, we know γ is not dense in any neighborhood of p, so we may choose some $c \notin \gamma$ sufficiently close to p. Set $T(x) = \mathrm{I}_{r,c}(x)$ and observe:

1. $\text{length}(T(\gamma)) = \text{length}(\gamma)$ from Condition (ii).
2. The maximum pointwise absolute curvature of $T(\gamma)$ is at least K, as the osculating circle at $T(q)$ has radius δ satisfying $\frac{1}{\delta} > K$.

So then T is indeed the desired map, completing the proof. □

Corollary 2.2. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a non-circular, C^2 closed curve. Then for any $K \in \mathbb{R}^+$ there exists a length-preserving and Möbius energy preserving transformation T such that $\max \kappa_{T(\gamma)}^{abs}(t) > K$.*

Proof. Since T is a Möbius transformation and $T(\gamma) \subseteq \mathbb{R}^n$, this result follows immediately from Theorem 1.1. □

We now calculate a formula for the pointwise absolute curvature of a inverted curve. This result will be useful in Section 3.

Proposition 2.3. *Let γ be a C^2 curve in $\mathbb{R}^n$ parametrized by arclength, and fix $c \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$. Let $\Delta(s) = |\gamma(s) - c|^2$ and $\tilde{\gamma} = I_{r,c}(\gamma)$. Then*

$$\kappa_{\tilde{\gamma}}^{abs}(s) = \frac{1}{r^2\Delta}\left|\gamma''\Delta^2 - \gamma'\Delta\Delta' - (\gamma - c)\Delta\Delta'' + (\gamma - c)(\Delta')^2\right|, \quad \forall s, \gamma(s) \neq c.$$

Proof. We omit input variables throughout the proof, so we may write

$$\begin{aligned}\tilde{\gamma} &= \frac{r^2}{\Delta}(\gamma - c), \quad \tilde{\gamma}' = \frac{r^2}{\Delta^2}(\gamma'\Delta - (\gamma - c)\Delta'), \quad \text{and} \\ \tilde{\gamma}'' &= \frac{r^2}{\Delta^3}\left(\gamma''\Delta^2 - (\gamma - c)\Delta\Delta'' - 2\gamma'\Delta\Delta' + 2(\gamma - c)(\Delta')^2\right).\end{aligned}$$

Define $\tilde{\nu}(t) = |\tilde{\gamma}'(t)|$. Then we can observe that

$$\begin{aligned}\tilde{\nu}^2 = \tilde{\gamma}' \bullet \tilde{\gamma}' &= \frac{r^4}{\Delta^4}(\gamma'\Delta - (\gamma - c)\Delta') \bullet (\gamma'\Delta - (\gamma - c)\Delta') \\ &= \frac{r^4}{\Delta^4}(\underbrace{\gamma' \bullet \gamma'}_{=1}\Delta^2 - \underbrace{2\gamma' \bullet (\gamma - c)}_{=\Delta'}\Delta\Delta' + \underbrace{(\gamma - c) \bullet (\gamma - c)}_{=\Delta}(\Delta')^2) = \frac{r^4}{\Delta^2}.\end{aligned}$$

Hence, we may conclude that $\tilde{\nu} = \frac{r^2}{\Delta}$, which implies $\tilde{\nu}' = \frac{-r^2\Delta'}{\Delta^2}$. As $\tilde{\gamma}$ will be regular, we can calculate that

$$\kappa_{\tilde{\gamma}}^{\text{abs}}(t) = \frac{1}{\tilde{\nu}^3}\left|\tilde{\nu}\tilde{\gamma}'' - \tilde{\nu}'\tilde{\gamma}'\right|.$$

Substitution of the above work then gives that

$$\begin{aligned}\kappa_{\tilde{\gamma}}^{\text{abs}} = \frac{\Delta^3}{r^6}\Bigg|&\frac{r^2}{\Delta}\left(\frac{r^2}{\Delta^3}\left(\gamma''\Delta^2 - (\gamma - c)\Delta\Delta'' - 2\gamma'\Delta\Delta' + 2(\gamma - c)(\Delta')^2\right)\right) \\ &+ \frac{r^2\Delta'}{\Delta^2}\left(\frac{r^2}{\Delta^2}(\gamma'\Delta - (\gamma - c)\Delta')\right)\Bigg|.\end{aligned}$$

This easily simplifies to the desired expression, so the proof is complete. □

3 Total Curvature

For a regular C^2 curve $\gamma : [0, \ell] \to \mathbb{R}^n$ with an arbitrary parametrization in t, we can calculate its total absolute curvature as

$$\kappa_{\text{tot}}^{\text{abs}}(\gamma) = \int_0^\ell |\kappa_\gamma^{\text{abs}}(t)||\gamma'(t)|\, dt.$$

Note that $\kappa_{\text{tot}}^{\text{abs}}(\gamma)$ is independent of the regular parametrization chosen. It can also be shown that

$$\kappa_{\text{tot}}^{\text{abs}}(A(\gamma)) = \kappa_{\text{tot}}^{\text{abs}}(\gamma)$$

for any affine similarity A of $\mathbb{R}^n$. This focuses the entirety of our interest on the effects of inversions.

Corollary 3.1. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a regular C^2 curve parametrized by arclength, let $r \in \mathbb{R}^+$, fix $c \in \mathbb{R}^n$, and let $\Delta = |\gamma(s) - c|^2$. Then for all $r \in \mathbb{R}^+$ we have*

(i) $\kappa_{tot}^{abs}(I_{r,c}(\gamma)) = \int_0^\ell \frac{1}{\Delta^2} |\gamma''\Delta^2 - \gamma'\Delta\Delta' - (\gamma - c)\Delta\Delta'' + (\gamma - c)(\Delta')^2| \, ds.$

(ii) $\kappa_{tot}^{abs}(I_{r,c}(\gamma)) = \kappa_{tot}^{abs}(I_{1,c}(\gamma)).$

(iii) $\kappa_{tot}^{abs}(I_{r,c}(\gamma))$ *is continuous in c when $c \notin \gamma$.*

Remark. The integral given in Part (i) of Corollary 3.1 is improper when $c \in \gamma$ and may be infinite, a priori. We show that it is finite in Proposition 3.9.

Proof. Part (i) follows from Proposition 2.3 and $|(\mathrm{I}_{r,c}(\gamma))'| = \frac{r^2}{\Delta}$, and Part (ii) is a consequence of Part (i). Notice $c \notin \gamma$ implies $\Delta > 0$, so the integrand of Part (i) is continuous when $c \notin \gamma$, which proves Part (iii). $\square$

Notice that the formula given in Corollary 3.1 has no dependence on r—this tells us that the total absolute curvature of an inverted curve is not affected by the radius of the inversion sphere. This narrows our focus purely to the effects of inversion centers. To acquire Proposition 3.18 and Theorem 3.19, we divide $\mathbb{R}^n$ into four regions, each which we analyze separately: points in a neighborhood of ∞, points on γ, points in a neighborhood of γ, and the remaining points.

3.1 Inversion centers in a neighborhood of infinity

Inversions through spheres with centers near infinity are "almost reflections" (see Figure 4).

Proposition 3.2. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a regular C^2 curve parametrized by arclength. Then*

$$\lim_{|c| \to \infty} \kappa_{tot}^{abs}(I_{r,c}(\gamma)) = \kappa_{tot}^{abs}(\gamma)$$

for any choice of r.

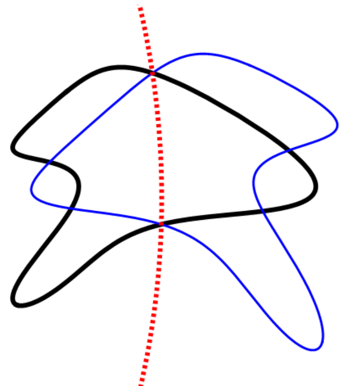

Figure 4: Inversion through a distant center is approximately a reflection.

Proof. Let $\epsilon > 0$, choose $R \in \mathbb{R}^+$ such that $\gamma \subseteq B^n_R(0)$, set $M = \frac{8\ell}{\epsilon} + R$, let $|c| > M$, and let $\tilde{\gamma} = \mathrm{I}_{r,c}(\gamma)$ for a chosen, fixed r. Using the result from Corollary 3.1 and the facts that

$$\Delta = (\gamma - c) \cdot (\gamma - c), \quad \Delta' = 2\gamma' \cdot (\gamma - c), \quad \text{and} \quad \Delta'' = 2 + 2\gamma'' \cdot (\gamma - c),$$

we can write the following:

$$\begin{aligned}
&|\kappa^{\text{abs}}_{\text{tot}}(\tilde{\gamma}) - \kappa^{\text{abs}}_{\text{tot}}(\gamma)| \\
&\leq \int_0^\ell \Bigg| \Bigg| \gamma'' - \gamma' \frac{2\gamma' \cdot (\gamma - c)}{|\gamma - c|^2} - (\gamma - c) \frac{2 + 2\gamma'' \cdot (\gamma - c)}{|\gamma - c|^2} \\
&\qquad\qquad + (\gamma - c) \frac{(2\gamma' \cdot (\gamma - c))^2}{|\gamma - c|^4} \Bigg| - |\gamma''| \Bigg| \, ds \\
&= \int_0^\ell \Bigg| \Bigg| \frac{1}{|\gamma - c|} \left[-2 \frac{(\gamma - c)}{|\gamma - c|} - 2\gamma' \left(\gamma' \cdot \frac{\gamma - c}{|\gamma - c|} \right) + 4 \frac{\gamma - c}{|\gamma - c|} \left(\gamma' \cdot \frac{\gamma - c}{|\gamma - c|} \right)^2 \right] \\
&\qquad\qquad + \left[\gamma'' - 2 \frac{\gamma'' \cdot (\gamma - c)}{(\gamma - c) \cdot (\gamma - c)} (\gamma - c) \right] \Bigg| - |\gamma''| \Bigg| \, ds
\end{aligned}$$

Recall that the reflection of a vector v in a hyperplane through the origin orthogonal to a vector a is given by the formula $\mathrm{Ref}_a(v) = v - 2\frac{v \cdot a}{a \cdot a} a$. Hence,

$$\gamma'' - 2 \frac{\gamma'' \cdot (\gamma - c)}{(\gamma - c) \cdot (\gamma - c)} (\gamma - c) = \mathrm{Ref}_{\gamma - c}(\gamma'').$$

In addition, note that $|\gamma''| = |\mathrm{Ref}_{\gamma-c}(\gamma'')|$. So we continue with the following:

$$\begin{aligned}
&|\kappa_{\text{tot}}^{\text{abs}}(\tilde{\gamma}) - \kappa_{\text{tot}}^{\text{abs}}(\gamma)| \\
&\leq \int_0^\ell \Bigg| \Bigg| \frac{1}{|\gamma - c|} \left[-2\frac{(\gamma - c)}{|\gamma - c|} - 2\gamma' \left(\gamma' \cdot \frac{\gamma - c}{|\gamma - c|}\right) + 4\frac{\gamma - c}{|\gamma - c|}\left(\gamma' \cdot \frac{\gamma - c}{|\gamma - c|}\right)^2 \right] \\
&\qquad\qquad + [\mathrm{Ref}_{\gamma-c}(\gamma'')] \Bigg| - |\mathrm{Ref}_{\gamma-c}(\gamma'')| \Bigg| \, ds \\
&\leq \int_0^\ell \frac{1}{|\gamma - c|} \left[2\left|\frac{(\gamma - c)}{|\gamma - c|}\right| + 2|\gamma'|^2 \left|\frac{\gamma - c}{|\gamma - c|}\right| + 4|\gamma'|^2 \left|\frac{\gamma - c}{|\gamma - c|}\right|^3 \right] + |0| \, ds \\
&= \int_0^\ell \frac{8}{|\gamma - c|} \, ds
\end{aligned}$$

Since $|c| > M$, we can then observe that

$$\int_0^\ell \frac{8}{|\gamma - c|} \, ds \leq \int_0^\ell \frac{8}{|c| - |\gamma|} \, ds \leq \int_0^\ell \frac{8}{|c| - R} \, ds = \frac{8\ell}{|c| - R} < \frac{8\ell}{M - R} = \epsilon.$$

Hence, we have that $\lim_{|c|\to\infty} \kappa_{\text{tot}}^{\text{abs}}(\tilde{\gamma}) = \kappa_{\text{tot}}^{\text{abs}}(\gamma)$, as desired. □

3.2 Inversion centers on the curve γ

Say that $c \in \gamma$, and notice that for this choice c of inversion center $\mathrm{I}_{r,c}(\gamma)$ is no longer be a closed curve in $\mathbb{R}^n$. Specifically, the image of the point $c \in \gamma$ is the point at infinity. We call such curve a *broken inverted curve*. Our goal in this subsection is Proposition 3.9.

Proposition 3.9. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. Then we have the following:*

(i) $\kappa_{tot}^{abs}(I_{1,\gamma(t)}(\gamma))$ is finite for all t.

(ii) The function $F : [0, \ell] \to \mathbb{R}$ defined by $F(t) = \kappa_{tot}^{abs}(I_{1,\gamma(t)}(\gamma))$ is uniformly continuous.

We first build up some preliminaries. Proposition 3.3 is a well-known result (for a recent exposition, see [6] and [2]), which we have reframed here in the context of our particular problem. If $c \in \mathbb{R}^n$, let $\mathrm{dist}(c, \gamma) = \inf_s |\gamma(s) - c|$.

Proposition 3.3. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. Then there exists some $d > 0$ such that for any $c \in \mathbb{R}^n$ satisfying $0 < dist(c, \gamma) < d$ we have the following:*

(i) There exists unique $t \in [0, \ell)$ such that $|\gamma(t) - c| = dist(c, \gamma)$.

(ii) There exists unique $\vec{\omega} \in T_{\gamma(t)}\mathbb{R}^n$ such that $|\vec{\omega}| = 1$, $\vec{\omega} \perp \gamma'(t)$, and $\vec{\omega}$ is in the same direction as $c - \gamma(t)$.

(iii) $c = \gamma(t) + dist(c, \gamma) \cdot \vec{\omega}$.

The value $r(\gamma) = \sup\{d \in \mathbb{R}^+ \mid d \text{ satisfies Proposition 3.3}\}$ is called the *normal injectivity radius of* γ, or thickness of γ. Lemma 3.4 is another well-known result (see [3]) needed to prove Proposition 3.9.

Lemma 3.4. *Let $h : (-\delta, \delta) \times U \to \mathbb{R}^n$ be a C^k function with $h(0, t) = 0$ for all $t \in U$. Then there exists a C^{k-1} function $g : (-\delta, \delta) \times U \to \mathbb{R}^n$ such that*

$$h(s, t) = sg(s, t) \quad \text{and} \quad g(0, t) = \frac{\partial h}{\partial s}(s, t)\Big|_{s=0}.$$

We now use Lemma 3.4 to construct a lemma critical to the main result of this subsubsection.

Lemma 3.5. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 closed curve and define $L : [-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell] \to \mathbb{R}^n$ by $L(s, t) = \gamma(s + t) - \gamma(t) - \gamma'(t)s - \frac{\gamma''(t)}{2}s^2$. Then*

$$\frac{L(s, t)}{s^2}, \quad \frac{\frac{\partial L}{\partial s}(s, t)}{s}, \quad \text{and} \quad \frac{\partial^2 L}{\partial s^2}(s, t)$$

are all continuously extendable to $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$.

Proof. Since γ is C^2, we immediately know that $L(s, t)$ is C^2 on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$, so $\frac{\partial^2 L}{\partial s^2}(s, t)$ is C^0 on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$. Notice for all t that $s = 0$ gives

$$L(0, t) = \gamma(0 + t) - \gamma(t) - \gamma'(t) \cdot 0 - \frac{\gamma''(t)}{2} \cdot 0^2 = 0.$$

Hence, Lemma 3.4 tells us that there exists some $G_1(s, t)$ such that G_1 is C^1 on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$ and $L(s, t) = sG_1(s, t)$. We can similarly obtain for all t that $s = 0$ gives $G_1(0, t) = 0$. Hence, there exists some $G_2(s, t)$ such that G_2 is C^0 on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$ and $G_1(s, t) = sG_2(s, t)$. As a result, we have for $s \neq 0$ that

$$L(s, t) = s^2 G_2(s, t) \implies \frac{L(s, t)}{s^2} = G_2(s, t).$$

Since $G_2(s,t)$ is C^0 on $[-\frac{\ell}{2},\frac{\ell}{2}]\times[0,\ell]$, $\frac{L(s,t)}{s^2}$ is extends continuously to $[-\frac{\ell}{2},\frac{\ell}{2}]\times[0,\ell]$. Similar to L and G_1 above, we can obtain for all t that $s=0$ gives $\frac{\partial L}{\partial s}(0,t)=0$. So then there exists some $G_3(s,t)$ such that K is C^0 on $[-\frac{\ell}{2},\frac{\ell}{2}]\times[0,\ell]$ and $\frac{\partial L}{\partial s}(s,t)=sG_3(s,t)$. As a result, we have for $s\neq 0$ that

$$\frac{\frac{\partial L}{\partial s}(s,t)}{s}=G_3(s,t).$$

Since $G_3=(s,t)$ is C^0 on $[-\frac{\ell}{2},\frac{\ell}{2}]\times[0,\ell]$, we know $\frac{\frac{\partial L}{\partial s}(s,t)}{s}$ is continuously extendable to $[-\frac{\ell}{2},\frac{\ell}{2}]\times[0,\ell]$. This completes the proof. $\square$

Lemma 3.6 is a basic result which allows us to rewrite rational expressions containing a function in a useful way.

Lemma 3.6. *Let $g(s)$ be a continuous function such that $1+sg(s)>0$ for all s in an open interval containing 0. Then it follows that there exists a function $\bar{g}$ such that $\bar{g}$ is continuous and satisfies*

$$\frac{1}{1+sg(s)}=1+s\bar{g}(s).$$

Lemma 3.7. *Let $\gamma:[0,\ell]\to\mathbb{R}^n$ be a regular C^2 simple closed curve parametrized by arclength and define $\Delta(x,y)=|\gamma(x)-\gamma(y)|^2$. Then*

$$\begin{aligned}f(s,t)=\Bigg|&\gamma''(s+t)-\gamma'(s+t)\frac{\Delta'(s+t,t)}{\Delta(s+t,t)}-(\gamma(s+t)-\gamma(t))\frac{\Delta''(s+t,t)}{\Delta(s+t,t)}\\&+(\gamma(s+t)-\gamma(t))\left(\frac{\Delta'(s+t,t)}{\Delta(s+t,t)}\right)^2\Bigg|\end{aligned}$$

is continuously extendable to $[-\frac{\ell}{2},\frac{\ell}{2}]\times[0,\ell]$, and it is bounded on that same set, where

$$\Delta'=\frac{\partial}{\partial s}\Delta\quad\textit{and}\quad\Delta''=\frac{\partial^2}{\partial s^2}\Delta.$$

Proof. Using L as defined in Lemma 3.5, we can write

$$\gamma(s+t)=\gamma(t)+\gamma'(t)s+\frac{\gamma''(t)}{2}s^2+L(s,t).$$

Now fix t, and set $B = \gamma'(t)$ and $C = \frac{\gamma''(t)}{2}$. We abbreviate the s-derivatives of L as $L'(s,t) = \frac{\partial L}{\partial s}(s,t)$ and $L''(s,t) = \frac{\partial^2 L}{\partial s^2}(s,t)$. Also omitting the input variables for L, we set $B = \gamma'(t)$ and $C = \frac{\gamma''(t)}{2}$ and write

$$\begin{aligned} \gamma(s+t) - \gamma(t) &= Bs + Cs^2 + L, \\ \gamma'(s+t) &= B + 2Cs + L', \quad \text{and} \\ \gamma''(s+t) &= 2C + L''. \end{aligned}$$

Let $\Delta = \Delta(s+t,t)$. Then we can also write out Δ and its s-derivatives below, which we abbreviate as Δ' and Δ''. As part of this process, we define W, U, and V as shown. In addition, we omit the input variables on Δ, L, and their derivatives. As γ is parametrized by arclength, $B \bullet B = 1$ and $B \bullet C = 0$. We then have the following for $s \neq 0$:

$$\begin{aligned} \Delta &= |\gamma(s+t) - \gamma(t)|^2 \\ &= B \bullet Bs^2 + 2B \bullet Cs^3 + 2B \bullet Ls + C \bullet Cs^4 + 2C \bullet Ls^2 + L \bullet L \\ &= s^2 + s^3 \left(2B \bullet \frac{L}{s^2} + C \bullet Cs + 2C \bullet \frac{L}{s} + \frac{L}{s^2} \bullet \frac{L}{s} \right) \end{aligned}$$

We know from Lemma 3.5 that $\frac{L}{s^2}$ is continuously extendable to $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$, so let $\tilde{L}$ be such a continuous extension. Note that this substitution also allows us to write $\frac{L}{s} = \tilde{L}s$. Then

$$\Delta = s^2 + s^3 \underbrace{\left(2B \bullet \tilde{L} + C \bullet Cs + 2C \bullet \tilde{L}s + \tilde{L} \bullet \tilde{L}s \right)}_{W} = s^2(1 + sW)$$

Note that all the terms of W are continuous on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$, so $(1 + sW)$ is continuous on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$, and we can continuously extend the above representation of Δ to $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$. Further, we know that $\Delta \geq 0$ for all s, with $\Delta = 0$ exactly when $s = 0$. This implies that $1 + sW > 0$ for all s. This will later permit the use of Lemma 3.6.

We apply similar ideas to calculate Δ' and Δ'', using $\hat{L}$ to represent the continuous extension of $\frac{L'}{s}$. We have the following for $s \neq 0$:

$$\begin{aligned} \Delta' &= 2\gamma'(s+t) \bullet (\gamma(s+t) - \gamma(t)) \\ &= 2s + 2s^2 \underbrace{\left(B \bullet \tilde{L} + B \bullet \hat{L} + 2C \bullet Cs + 2C \bullet \tilde{L}s + C \bullet L' + \tilde{L} \bullet L' \right)}_{U} \\ &= 2s(1 + sU) \end{aligned}$$

We can also calculate the following for $s \neq 0$:

$$\begin{aligned}
\Delta'' &= 2\gamma''(s+t) \bullet (\gamma(s+t) - \gamma(t)) + 2\gamma'(s+t) \bullet \gamma'(s+t) \\
&= 2 + 2s \underbrace{\left(2C \bullet Cs + 2C \bullet \tilde{L}s + B \bullet L'' + C \bullet L''s + \tilde{L}s \bullet L''\right)}_{V} \\
&= 2(1 + sV)
\end{aligned}$$

Similarly to Δ, these representations of Δ' and Δ'' continuously extend to $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$. By Lemma 3.6 there exists a continuous function $\bar{W}$ satisfying

$$\frac{1}{1 + sW} = 1 + s\bar{W}.$$

Hence, we can write the following, with X, Y, and Z defined as shown:

$$\begin{aligned}
\frac{\Delta'}{\Delta} &= \frac{2s(1+sU)}{s^2(1+sW)} = \frac{2}{s}\big(1 + s\underbrace{(U + \bar{W} + sU\bar{W})}_{X}\big) = \frac{2}{s}(1 + sX) \\
\frac{\Delta''}{\Delta} &= \frac{2(1+sV)}{s^2(1+sW)} = \frac{2}{s^2}\big(1 + s\underbrace{(V + \bar{W} + sV\bar{W})}_{Y}\big) = \frac{2}{s^2}(1 + sY) \\
\left(\frac{\Delta'}{\Delta}\right)^2 &= \left(\frac{2}{s}(1+sX)\right)^2 = \frac{4}{s^2}\big(1 + s\underbrace{(2X + sX^2)}_{Z}\big) = \frac{4}{s^2}(1 + sZ)
\end{aligned}$$

Note that the continuity of U, V, and $\bar{W}$ on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$ ensures that X, Y, and Z are continuous on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$. The above setup now allows us to write the following for $s \neq 0$:

$$
\begin{aligned}
f(s,t) &= \bigg| 2C + L'' - (B + 2Cs + L') \cdot \frac{2}{s}(1 + sX) \\
&\qquad - (Bs + Cs^2 + L) \cdot \frac{2}{s^2}(1 + sY) + (Bs + Cs^2 + L) \cdot \frac{4}{s^2}(1 + sZ) \bigg| \\
&= \bigg| 4\frac{B}{s} - 2\frac{B}{s} - 2\frac{B}{s} + 2C - 2C + 4C - 4C + 4\tilde{L} - 2\tilde{L} - 2\hat{L} + L'' \\
&\qquad + X\left(-2B - 4Cs - 2\hat{L}s\right) + Y\left(-2B - 2Cs - 2\tilde{L}s\right) \\
&\qquad + Z\left(4B + 4Cs + 4\tilde{L}s\right) \bigg| \\
&= \bigg| 2\tilde{L} - 2\hat{L} + L'' + X\left(-2B - 4Cs - 2\hat{L}s\right) \\
&\qquad + Y\left(-2B - 2Cs - 2\tilde{L}s\right) + Z\left(4B + 4Cs + 4\tilde{L}s\right) \bigg|
\end{aligned}
$$

Note that each of the terms in the above representation of f is continuous on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$, so we may continuously extend f to $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$. The crucial point here is the cancellation of the terms containing $\frac{B}{s}$. As this extension is continuous on a compact set, it is also bounded. $\square$

Lemma 3.8 is a basic and standard fact about integration.

Lemma 3.8. *Let $f : [-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell] \to \mathbb{R}$ be continuous. Then it follows that*

$$F(t) = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} f(s,t)\, ds$$

is a continuous function in t.

We finally turn our attention to the proof of Proposition 3.9.

Proposition 3.9. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. Then we have the following:*

(i) $\kappa_{tot}^{abs}(I_{1,\gamma(t)}(\gamma))$ is finite for all t.

(ii) The function $F : [0, \ell] \to \mathbb{R}$ defined by $F(t) = \kappa_{tot}^{abs}(I_{1,\gamma(t)}(\gamma))$ is uniformly continuous.

Proof. Note that the function in Lemma 3.7 can be integrated in s over $[0, \ell]$ to obtain $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{1,\gamma(t)}(\gamma))$. Then combining the results from Lemmas 3.7 and 3.8 and Corollary 3.1 gives us both finiteness and continuity. As the domain of F is $[0, \ell]$ (a compact set), it follows that the continuity of F is uniform. $\square$

3.3 Points in a neighborhood of the curve γ

Of the four regions, this one requires the most careful handling. Our work here builds toward a proof of Proposition 3.15, which describes the behavior of the total absolute curvature as the center of inversion approaches the curve.

Proposition 3.15. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. For all $\epsilon > 0$ there exists $d > 0$ for all $t \in [0, \ell]$ and $\vec{\omega}$ satisfying $|\vec{\omega}| = 1$ and $\vec{\omega} \perp \gamma'(t)$ such that*

$$\left|\kappa_{tot}^{abs}(I_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{tot}^{abs}(I_{r,\lambda(u)}(\gamma))\right| < \epsilon,$$

whenever $0 < u < d$, where $\lambda(u) = \gamma(t) + u\vec{\omega}$.

This result is obtained in three main steps: First, we must quantify the behavior of the total absolute curvature coming from points of γ near the inversion point (we will show that it is approximately equal to 2π). Second, we must quantify the behavior of the total absolute curvature coming from the points of γ away from the inversion point (we will show it is approximately equal to the curvature of a broken inverted curve). Finally, we will combine these estimates to derive an estimate for the total absolute curvature of the entirety of the inverted image of γ. Note we also require γ to be closed.

To understand the curvature of the inverted curve when the inversion center c is nearby (but not on) the curve γ, we use the following strategy (illustrated in Figure 5):

1. For a point c sufficiently close to the curve γ, let t be such that $\gamma(t)$ is the unique point on γ closest to c.
2. Specify a line λ through c and $\gamma(t)$ parametrized by arclength in u.
3. Specify cut points $\gamma(t + \delta(u))$ and $\gamma(t - \delta(u))$ separating γ into two segments—an interval about $\gamma(t)$ and its complement.
4. For a fixed c set u so that $\lambda(u) = c$, then $u = \mathrm{dist}(c, \gamma)$. Notice the cut points given by $\delta(u)$ then depend on $\mathrm{dist}(c, \gamma)$. Specifically, we will define δ so that $\delta(u) \to 0$ as $\mathrm{dist}(c, \gamma) \to 0$.

5. Analyze the total absolute curvature of each segment separately, then combine the two results.

The choice of $\delta(u)$ only affects the total absolute curvatures of the individual segments, not the entire curve.

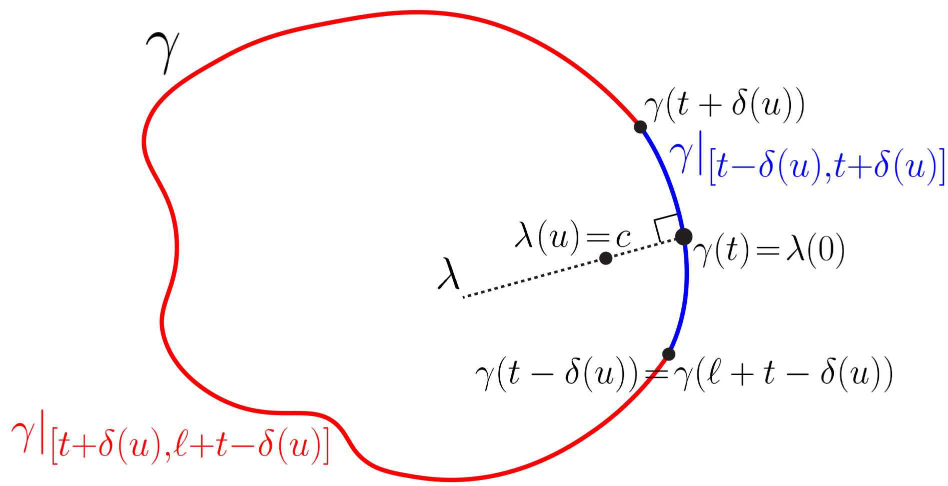

Figure 5: An approach for analyzing the total absolute curvature of γ when the inversion center is close to γ.

We will prove an array of lemmas. The first is Lemma 3.10, which effectively says an inversion with an inversion center very close to a circular arc transforms the arc into circle minus an a small arc.

Lemma 3.10. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. For all $\epsilon > 0$ there exists $d > 0$ such that for all $t \in [0, \ell]$ and $\vec{\omega}$ satisfying $|\vec{\omega}| = 1$ and $\vec{\omega} \perp \gamma'(t)$ one has*

$$\left|\kappa^{abs}_{tot}(I_{r,\lambda(u)}(\eta_t|_{[-\delta(u),\delta(u)]})) - 2\pi\right| < \epsilon$$

whenever $0 < u < d$, where $\lambda(u) = \gamma(t) + u\vec{\omega}$, $\delta(u) = u^{\frac{1}{9}}$, and η_t is the osculating circle of γ at $\gamma(t)$. (Note that η_t will be a line in the case that $\kappa^{abs}_{\gamma}(t) = 0$.)

Proof. Let $\epsilon > 0$ and fix t and u. We have two cases.

Case 1: η_t is a circle with radius r. We abbreviate η_t as η. Since total curvature is invariant with respect to affine similarities, we may say without loss of generality that $\eta(s) = (\sin(s), \cos(s) - 1, 0, \dots, 0)$. We then have that $\eta'(0) = (1, 0, 0, \dots, 0)$, so we know $\vec{\omega}$ is of the form $\vec{\omega} = \frac{1}{r}(0, \omega_2, \omega_3, \dots, \omega_n)$. (Note that rescaling the circle to ensure that it has radius one also necessarily scales $\vec{\omega}$.) We can compute the following:

$$\begin{aligned}
\eta(s) - \lambda(u) &= \left(\sin(s), \cos(s) - 1 - u\frac{\omega_2}{r}, -u\frac{\omega_3}{r}, \dots, -u\frac{\omega_n}{r}\right) \\
\eta'(s) &= (\cos(s), -\sin(s), 0, \dots, 0) \\
\eta''(s) &= (-\sin(s), \cos(s), 0, \dots, 0) \\
\Delta(s,u) = |\eta(s) - \lambda(u)|^2 &= 2 + 2u\frac{\omega_2}{r} - 2\left(1 + u\frac{\omega_2}{r}\right)\cos(s) + \frac{u^2}{r^2} \\
\Delta'(s,u) &= 2\left(1 + u\frac{\omega_2}{r}\right)\sin(s) \\
\Delta''(s,u) &= 2\left(1 + u\frac{\omega_2}{r}\right)\cos(s)
\end{aligned}$$

We substitute the above work into Corollary 3.1 to obtain the following:

$$\begin{aligned}
\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\eta|_{[-\delta(u),\delta(u)]})) &= \int_{-\frac{\delta(u)}{r}}^{\frac{\delta(u)}{r}} \frac{u\sqrt{4r^2 + 4ru\omega_2 + u^2}}{2r^2 + 2ru\omega_2 + u^2 - 2r(r + u\omega_2)\cos(s)}\, ds \\
&= 4\arctan\left[\frac{\tan\left(\frac{\delta(u)}{2r}\right)\sqrt{4r^2 + 4ru\omega_2 + u^2}}{u}\right]
\end{aligned}$$

As γ is compact, $\max_s \kappa_\gamma^{\text{abs}}(s)$ exists; call it $\kappa_m > 0$, and note that $r \geq \frac{1}{\kappa_m}$ for every osculating circle η_t. Note that $\tan(x) \geq x$ for sufficiently small $0 \leq x \leq \frac{\pi}{2}$, so $\tan\left(\frac{\delta(u)}{2r}\right) \geq \frac{\delta(u)}{2r}$ for sufficiently small $u \geq 0$. Finally, note that $|\omega_2| \leq 1$. Then for $u \leq \frac{1}{\kappa_m}$ we have

$$\frac{\tan\left(\frac{\delta(u)}{2r}\right)\sqrt{4r^2 + 4ru\omega_2 + u^2}}{u} \geq \frac{1}{u^{\frac{8}{9}}}\sqrt{1 + \frac{u}{r}\omega_2 + \frac{u^2}{4r^2}} \geq \frac{1}{u^{\frac{8}{9}}}\left(1 - \frac{u}{2r}\right).$$

Note that $u \leq \frac{1}{\kappa_m} \leq r$, so we can obtain that $1 - \frac{u}{2r} \geq \frac{1}{2}$. Hence,

$$\frac{\tan\left(\frac{\delta(u)}{2r}\right)\sqrt{4r^2 + 4ru\omega_2 + u^2}}{u} \geq \frac{1}{u^{\frac{8}{9}}} \cdot \frac{1}{2} = \frac{1}{2u^{\frac{8}{9}}}.$$

Notice that this reduction removes any dependence on t. Recall that arctan is increasing and bounded above by $\frac{\pi}{2}$. We then have the following:

$$\begin{aligned}
&|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\eta|_{[-\delta(u),\delta(u)]})) - 2\pi| \\
&= \left|4\arctan\left[\frac{\tan\left(\frac{\delta(u)}{2}\right)\sqrt{4r^2+4ru\omega_2+u^2}}{u}\right] - 2\pi\right| \\
&\leq \left|4\arctan\left[\frac{1}{2u^{\frac{8}{9}}}\right] - 2\pi\right|
\end{aligned}$$

Then for all t where η_t is a circle, there exists some $d_1 > 0$ such that

$$\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\eta|_{[-\delta(u),\delta(u)]})) - 2\pi\right| \leq \left|4\arctan\left[\frac{1}{2u^{\frac{8}{9}}}\right] - 2\pi\right| < \epsilon$$

when $0 < u < d_1$.

Case 2: η is a line. We can follow a process similar to that shown in Case 1 to obtain for all t where η_t is a line that there exists some $d_2 > 0$ such that

$$\begin{aligned}
\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\eta|_{[-\delta(u),\delta(u)]})) - 2\pi\right| &= \left|\int_{-\delta(u)}^{\delta(u)} \frac{2u}{s^2+u^2}\, ds - 2\pi\right| \\
&= \left|4\arctan\left[\frac{1}{u^{\frac{8}{9}}}\right] - 2\pi\right| < \epsilon
\end{aligned}$$

when $0 < u < d_2$. Picking $d = \min\{d_1, d_2\}$ completes the proof. $\square$

Lemma 3.11 essentially the same as Lemma 3.5, so we omit its proof.

Lemma 3.11. *Let $\gamma : [0,\ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength and for $t \in [0,\ell]$ let σ_t be its parabolic approximation at $\gamma(t)$. That is $\sigma_t(s) = \gamma(t) + \gamma'(t)s + \frac{1}{2}\gamma''(t)s^2$. Now define $L_t : [0,\ell] \to \mathbb{R}^n$ by $L_t(s) = \gamma(s+t) - \sigma_t(s)$. Then*

$$\frac{L_t(s)}{s^2}, \quad \frac{L_t'(s)}{s}, \quad \textit{and} \quad L_t''(s)$$

are all continuously extendable to $[-\frac{\ell}{2}, \frac{\ell}{2}]$ for all t and bounded uniformly on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0,\ell]$.

Lemma 3.12 says that an inverted segment of the parabolic approximation of a curve γ at a point provides an approximation of the total absolute curvature of an inverted segment of γ at that point for a sufficiently close inversion center and sufficiently small segments.

Lemma 3.12. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. For all $\epsilon > 0$ there exists $d > 0$ such that for all $t \in [0, \ell]$ and $\vec{\omega}$ satisfying $|\vec{\omega}| = 1$ and $\vec{\omega} \perp \gamma'(t)$ one has*

$$\left|\kappa_{tot}^{abs}(I_{r,\lambda(u)}(\gamma|_{[t-\delta(u),t+\delta(u)]})) - \kappa_{tot}^{abs}(I_{r,\lambda(u)}(\sigma_t|_{[-\delta(u),\delta(u)]}))\right| < \epsilon$$

whenever $0 < u < d$, where $\lambda(u) = \gamma(t) + u\vec{\omega}$, $\delta(u) = u^{\frac{1}{9}}$, and

$$\sigma_t(s) = \gamma(t) + s\gamma'(t) + \frac{s^2}{2}\gamma''(t)$$

is the parabolic approximation of γ at $\gamma(t)$, with $s \in [-\frac{\ell}{2}, \frac{\ell}{2}]$.

Proof. Let $\epsilon > 0$. We first prove the case where $t = 0$, then show that the proof for $t = 0$ can be extended to work for any t. That said, set $t = 0$, and abbreviate σ_t as σ. Let $\Delta_\gamma(s, u) = |\gamma(s) - \lambda(u)|^2$ and $\Delta_\sigma(s, u) = |\sigma(s) - \lambda(u)|^2$. Fix u and set $\Delta' = \frac{\partial \Delta}{\partial s}$. We begin with the following, omitting all input variables and defining N_i and D as shown below:

$$\begin{aligned}
&\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[-\delta,\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]}))\right| \\
&= \left| \int_{-\delta}^{\delta} \left| \gamma'' - \gamma' \frac{\Delta'_\gamma}{\Delta_\gamma} - (\gamma - \lambda)\frac{\Delta''_\gamma}{\Delta_\gamma} + (\gamma - \lambda)\frac{(\Delta'_\gamma)^2}{\Delta_\gamma^2} \right| ds \right. \\
&\qquad \left. - \int_{-\delta}^{\delta} \left| \sigma'' - \sigma' \frac{\Delta'_\sigma}{\Delta_\sigma} - (\sigma - \lambda)\frac{\Delta''_\sigma}{\Delta_\sigma} + (\sigma - \lambda)\frac{(\Delta'_\sigma)^2}{\Delta_\sigma^2} \right| ds \right| \\
&\le \int_{-\delta}^{\delta} |\underbrace{\gamma'' - \sigma''}_{N_1}|\, ds \\
&\quad + \int_{-\delta}^{\delta} \Bigg[\Big| - \underbrace{\left(\gamma'\Delta'_\gamma\Delta_\gamma\Delta_\sigma^2 - \sigma'\Delta'_\sigma\Delta_\sigma\Delta_\gamma^2\right)}_{N_2} \\
&\qquad - \underbrace{\left((\gamma - \lambda)\Delta''_\gamma\Delta_\gamma\Delta_\sigma^2 - (\sigma - \lambda)\Delta''_\sigma\Delta_\sigma\Delta_\gamma^2\right)}_{N_3} \\
&\qquad + \underbrace{\left((\gamma - \lambda)(\Delta'_\gamma)^2\Delta_\sigma^2 - (\sigma - \lambda)(\Delta'_\sigma)^2\Delta_\gamma^2\right)}_{N_4} \Big| / \underbrace{(\Delta_\gamma^2\Delta_\sigma^2)}_{D} \Bigg] ds
\end{aligned} \tag{1}$$

Over the remainder of this proof we shall derive individual bounds on N_1, N_2, N_3, N_4, and D to show that

$$\int_{-\delta}^{\delta} |N_1|\, ds + \int_{-\delta}^{\delta} \frac{|-N_2 - N_3 + N_4|}{D}\, ds \leq 2Mu^{\frac{1}{9}} + \frac{18Mu}{u^{\frac{8}{9}} - 12Mu}. \tag{2}$$

Define $L : [-\frac{\ell}{2}, \frac{\ell}{2}] \to \mathbb{R}^n$ by $L(s) = \gamma(s) - \sigma(s)$, which implies that $\gamma(s) = \sigma(s) + L(s)$. Defining $A = \gamma(0)$, $B = \gamma'(0)$, and $C = \frac{\gamma''(0)}{2}$, we may write $\sigma(s) = A + Bs + Cs^2$, as well as

$$\gamma(s) - \lambda(u) = Bs + Cs^2 + L - u\vec{\omega} \quad \text{and} \quad \sigma(s) - \lambda(u) = Bs + Cs^2 - u\vec{\omega}.$$

We can also compute

$$\gamma'(s) = B + 2Cs + L', \quad \sigma'(s) = B + 2Cs, \quad \gamma''(s) = 2C + L'', \quad \text{and} \quad \sigma''(s) = 2C.$$

As γ is parametrized by arclength, $B \bullet C = 0$ and $B \bullet B = 1$. In addition, our choice of ω implies $B \bullet \vec{\omega} = 0$ and $\vec{\omega} \bullet \vec{\omega} = 1$. We can then write the following:

$$\begin{aligned}
\Delta_\gamma &= s^2 + u^2 + s(2B \bullet L) + s^4(C \bullet C) + s^2(2C \bullet L) + L \bullet L \\
&\quad - s^2u(2C \bullet \vec{\omega}) - u(2L \bullet \vec{\omega}) \\
\Delta'_\gamma &= s \cdot 2 + 2B \bullet L + s(2B \bullet L') + s^3(4C \bullet C) \\
&\quad + s(4C \bullet L) + s^2(2C \bullet L') + 2L \bullet L' \\
&\quad - su(4C \bullet \vec{\omega}) - u(2L' \bullet \vec{\omega}) \\
\Delta''_\gamma &= 2 + 4B \bullet L' + s(2B \bullet L'') + s^2(12C \bullet C) \\
&\quad + 4C \bullet L + s(8C \bullet L') + s^2(2C \bullet L'') + 2L \bullet L'' + 2L' \bullet L' \\
&\quad - u(4C \bullet \vec{\omega}) - u(2L'' \bullet \vec{\omega})
\end{aligned}$$

We know from Lemma 3.11 that $\frac{L}{s^2}$ and $\frac{L'}{s}$ are continuously extendable to $[-\frac{\ell}{2}, \frac{\ell}{2}]$. Call these extensions $\tilde{L}$ and $\hat{L}$, respectively, and note that then $s^2\tilde{L} = L$ and $s\hat{L} = L'$. We then rewrite γ, σ, and their derivatives as follows:

$$\begin{aligned}
\gamma(s) - \lambda(u) &= sB + s^2C + s^2\tilde{L} - u\vec{\omega} & \sigma(s) - \lambda(u) &= sB + s^2C - u\vec{\omega} \\
\gamma'(s) &= B + s2C + s\hat{L} & \sigma'(s) &= B + s2C \\
\gamma''(s) &= 2C + L'' & \sigma''(s) &= 2C
\end{aligned}$$

We also rewrite Δ_γ and its derivatives as follows:

$$\begin{aligned}
\Delta_\gamma &= s^2 + u^2 + s^3(2B \bullet \tilde{L}) + s^4(C \bullet C) + s^4(2C \bullet \tilde{L}) + s^4(\tilde{L} \bullet \tilde{L}) \\
&\quad - s^2u(2C \bullet \vec{\omega}) - s^2u(2\tilde{L} \bullet \vec{\omega}) \\
\Delta'_\gamma &= s \cdot 2 + s^2(2B \bullet \hat{L}) + s^2(2B \bullet \tilde{L}) + s^3(4C \bullet C) \\
&\quad + s^3(4C \bullet \tilde{L}) + s^3(2C \bullet \hat{L}) + s^3(2\tilde{L} \bullet \hat{L}) \\
&\quad - su(4C \bullet \vec{\omega}) - su(2\hat{L} \bullet \vec{\omega}) \\
\Delta''_\gamma &= 2 + s(4B \bullet \hat{L}) + s(2B \bullet L'') + s^2(12C \bullet C) \\
&\quad + s^2(4C \bullet \tilde{L}) + s^2(8C \bullet \hat{L}) + s^2(2C \bullet L'') + s^2(2\tilde{L} \bullet L'') + s^2(2\hat{L} \bullet \hat{L}) \\
&\quad - u(4C \bullet \vec{\omega}) - u(2L'' \bullet \vec{\omega})
\end{aligned}$$

We can make similar calculations for σ.

$$\begin{aligned}
\Delta_\sigma &= s^2 + u^2 + s^4(C \bullet C) - s^2u(2C \bullet \vec{\omega}) \\
\Delta'_\sigma &= s \cdot 2 + s^3(4C \bullet C) - su(4C \bullet \vec{\omega}) \\
\Delta''_\sigma &= 2 + s^2(12C \bullet C) - u(4C \bullet \vec{\omega})
\end{aligned}$$

In order to simplify our calculations, we will use the following consolidated versions of each of these Δ-terms, with the $X_{i,j}, Y_{i,j}$ being defined appropriately. Note that while the $X_{i,j}, Y_{i,j}$ might contain variables s and u, they can each be bounded for all s and u when $t = 0$. (In fact, notice that they can actually be uniformly bounded for all t. This will be used later when extending the proof of the $t = 0$ case to the general case.)

$$\begin{aligned}
\Delta_\gamma &= s^2 + u^2 + s^3X_{1,1} + s^2uX_{1,2} & \Delta_\sigma &= s^2 + u^2 + s^3Y_{1,1} + s^2uY_{1,2} \\
\Delta'_\gamma &= s \cdot 2 + s^2X_{1,3} + suX_{1,4} & \Delta'_\sigma &= s \cdot 2 + s^2Y_{1,3} + suY_{1,4} \\
\Delta''_\gamma &= 2 + sX_{1,5} + uX_{1,6} & \Delta''_\sigma &= 2 + sY_{1,5} + uY_{1,6}
\end{aligned}$$

Now observe for appropriately defined $X_{2,j}, Y_{2,j}$ that

$$\Delta_\gamma^2 = (s^2 + u^2)^2 + \sum_{j=2}^{5} s^j u^{5-j} X_{2,j} \quad \text{and} \quad \Delta_\sigma^2 = (s^2 + u^2)^2 + \sum_{j=2}^{5} s^j u^{5-j} Y_{2,j}.$$

We now proceed to the calculation of the N_i, starting with N_1. Observe that

$$N_1 = \gamma'' - \sigma'' = L'' \implies \boxed{N_1 = L''}.$$

We now focus our efforts on the calculation each of the remaining cross terms in Expression (1), appropriately defining all $X_{i,j}, Y_{i,j}$ as the calculation proceeds. We begin by calculating the factors of N_2.

$$\Delta'_\gamma \Delta_\gamma = 2s(s^2+u^2) + \sum_{j=1}^{4} s^j u^{4-j} X_{3,j}$$

$$\Delta'_\sigma \Delta_\sigma = 2s(s^2+u^2) + \sum_{j=1}^{4} s^j u^{4-j} Y_{3,j}$$

Hence, we can obtain the following:

$$\Delta'_\gamma \Delta_\gamma \Delta_\sigma^2 = 2s(s^2+u^2)^3 + \sum_{j=1}^{8} s^j u^{8-j} X_{4,j}$$

$$\Delta'_\sigma \Delta_\sigma \Delta_\gamma^2 = 2s(s^2+u^2)^3 + \sum_{j=1}^{8} s^j u^{8-j} Y_{4,j}$$

We then multiply by γ' and σ', respectively; as well, we also define $V_{i,j}$ and $U_{i,j}$ appropriately. Unlike $X_{i,j}$ and $Y_{i,j}$, the $V_{i,j}$ and $U_{i,j}$ will be vector quantities, but as with those definitions above, $V_{i,j}$ and $U_{i,j}$ will still be bounded uniformly in s and u for $t = 0$. Recall $\gamma' = B + 2sC + \hat{L}s$.

$$\gamma' \Delta'_\gamma \Delta_\gamma \Delta_\sigma^2 = 2sB(s^2+u^2)^3 + \sum_{j=1}^{8} s^j u^{8-j} U_{1,j}$$

$$\sigma' \Delta'_\sigma \Delta_\sigma \Delta_\gamma^2 = 2sB(s^2+u^2)^3 + \sum_{j=1}^{8} s^j u^{8-j} V_{1,j}$$

Recall that N_2 is defined as the difference of these two terms. Then by defining appropriate $T_{i,j}$, which, again, we know can all be bounded uniformly in s and u for $t = 0$ we have that

$$\boxed{N_2 = \sum_{j=1}^{8} s^j u^{8-j} T_{1,j}}\,.$$

We now embark on a similar calculation of N_3, starting with its factors.

$$\Delta''_\gamma \Delta_\gamma = 2(s^2+u^2) + \sum_{j=0}^{3} s^j u^{3-j} X_{5,j}$$

$$\Delta''_\sigma \Delta_\sigma = 2(s^2+u^2) + \sum_{j=0}^{3} s^j u^{3-j} Y_{5,j}$$

Hence, we can obtain the following:

$$\Delta''_\gamma \Delta_\gamma \Delta_\sigma^2 = 2(s^2+u^2)^3 + \sum_{j=0}^{7} s^j u^{7-j} X_{6,j}$$

$$\Delta''_\sigma \Delta_\sigma \Delta_\gamma^2 = 2(s^2+u^2)^3 + \sum_{j=0}^{7} s^j u^{7-j} Y_{6,j}$$

We then multiply by $\gamma - \lambda$ and $\sigma - \lambda$, respectively.

$$(\gamma-\lambda)\Delta''_\gamma \Delta_\gamma \Delta_\sigma^2 = (2sB - 2u\vec{\omega})(s^2+u^2)^3 + \sum_{j=0}^{8} s^j u^{8-j} U_{2,j}$$

$$(\sigma-\lambda)\Delta''_\sigma \Delta_\sigma \Delta_\gamma^2 = (2sB - 2u\vec{\omega})(s^2+u^2)^3 + \sum_{j=0}^{8} s^j u^{8-j} V_{2,j}$$

Recall that N_3 is defined as the difference of these two terms, so then

$$\boxed{N_3 = \sum_{j=0}^{8} s^j u^{8-j} T_{2,j}}.$$

To calculate N_4, we first obtain the following:

$$(\Delta'_\gamma)^2 \Delta_\sigma^2 = 4s^2(s^2+u^2)^2 + \sum_{j=2}^{7} s^j u^{7-j} X_{7,j}$$

$$(\Delta'_\sigma)^2 \Delta_\gamma^2 = 4s^2(s^2+u^2)^2 + \sum_{j=2}^{7} s^j u^{7-j} Y_{7,j}$$

Multiply by $\gamma - \lambda$ and $\sigma - \lambda$, respectively.

$$(\gamma - \lambda)(\Delta'_\gamma)^2\Delta_\sigma^2 = 4s^2(sB - u\vec{\omega})(s^2+u^2)^2 + \sum_{j=2}^{8} s^j u^{8-j} U_{3,j}$$
$$(\sigma - \lambda)(\Delta'_\sigma)^2\Delta_\gamma^2 = 4s^2(sB - u\vec{\omega})(s^2+u^2)^2 + \sum_{j=2}^{8} s^j u^{8-j} V_{3,j}$$

As the difference of these two terms is N_4, we have

$$\boxed{N_4 = \sum_{j=2}^{8} s^j u^{8-j} T_{3,j}}.$$

We can then combine N_2, N_3, and N_4 to obtain the numerator of the second integrand in Expression (1).

$$-N_2 - N_3 + N_4 = \sum_{j=0}^{8} s^j u^{8-j} T_{4,j}$$

Similar to each of the N_i, we can calculate D as

$$\boxed{D = \Delta_\gamma^2\Delta_\sigma^2 = (s^2+u^2)^4 + \sum_{j=2}^{9} s^j u^{9-j} T_{5,j}}.$$

We shall now proceed to determining a bound for $|-N_2 - N_3 + N_4|/D$. As each of the $T_{i,j}$ are finite products and sums of bounded terms, we know there exists some $M \in \mathbb{R}^+$ such that $M \geq |T_i|$ for all i, j when $t = 0$ and $M \geq 6$. Hence, we have that

$$|-N_2 - N_3 + N_4| \leq M \sum_{j=0}^{8} |s|^j u^{8-j}.$$

Our bounds of integration then tell us that $|s| \leq \delta$, but we also have that $\delta = u^{\frac{1}{9}}$, which gives the following for sufficiently small u:

$$|-N_2 - N_3 + N_4| \leq M \sum_{j=0}^{8} (u^{\frac{1}{9}})^j u^{8-j} \leq 9Mu^{\frac{8}{9}}$$

Since $|s| \leq \delta = u^{\frac{1}{9}}$, for sufficiently small $u > 0$ we also have the following:

$$D \geq (u^{\frac{1}{9}})^8 - M\left(\sum_{j=0}^{3}(u^{\frac{1}{9}})^{2j}u^{8-2j} + \sum_{j=2}^{9}(u^{\frac{1}{9}})^j u^{9-j}\right) \geq u^{\frac{8}{9}} - 12Mu > 0$$

This allows us to bound the second integral in Expression (1). Also recalling that $\delta = u^{\frac{1}{9}}$, we can obtain

$$\int_{-\delta}^{\delta} \frac{|-N_2 - N_3 + N_4|}{D}\, ds \leq \int_{-\delta}^{\delta} \frac{9Mu^{\frac{8}{9}}}{u^{\frac{8}{9}} - 12Mu}\, ds = \frac{18Mu}{u^{\frac{8}{9}} - 12Mu}.$$

Ensure that $M \geq |L''|$. Then

$$\int_{-\delta}^{\delta} |N_1|\, ds = \int_{-\delta}^{\delta} |L''|\, ds \leq \int_{-\delta}^{\delta} M\, ds = 2Mu^{\frac{1}{9}}.$$

Combining these two integral bounds then gives us that

$$\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[-\delta,\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]}))\right| \leq 2Mu^{\frac{1}{9}} + \frac{18Mu}{u^{\frac{8}{9}} - 12Mu}.$$

Note that this is exactly the bound specified in Inequality (2), so our claim is proved. It then follows that there exists some $d > 0$ such that

$$2Mu^{\frac{1}{9}} + \frac{18Mu}{u^{\frac{8}{9}} - 12Mu} < \epsilon$$

whenever $0 < u < d$ for $t = 0$. Hence, we have that

$$\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[-\delta,\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]}))\right| < \epsilon$$

whenever $0 < u < d$ for $t = 0$, as desired. This completes the proof for $t = 0$.

This proof can be extended for all t. First, notice that the bounds above are only written in terms of M and u. As u is obviously independent of t, it remains only to show that our choice of M is independent of of t. Recall that M was chosen so that $M \geq |T_{i,j}|$ and $M \geq |L''|$. Each of these terms is comprised of the variables s and u and terms coming from γ (specifically, A, B, C, $\tilde{L}$, $\hat{L}$, and L''). s and u are on compact domains, and A, B, and C depend only on t. As the domain of γ is compact and γ is C^2, M can be chosen independently of all of these. The functions $\tilde{L}$, $\hat{L}$, and L'' are all continuous on the square $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$, so then M may also be chosen independently of them. Thus, we can choose some M entirely independent of t, and our proof of the $t = 0$ case uniformly extends to all choices of t. $\square$

Lemma 3.13 combines Lemmas 3.10 and 3.12 to give us a result similar to Lemma 3.10 for the general curve γ.

Lemma 3.13. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. For all $\epsilon > 0$ there exists $d > 0$ such that for all $t \in [0, \ell]$ and $\vec{\omega}$ satisfying $|\vec{\omega}| = 1$ and $\vec{\omega} \perp \gamma'(t)$ one has*

$$\left|\kappa_{tot}^{abs}(I_{r,\lambda(u)}(\gamma|_{[t-\delta(u),t+\delta(u)]})) - 2\pi\right| < \epsilon$$

whenever $0 < u < d$, where $\lambda(u) = \gamma(t) + u\vec{\omega}$ and $\delta(u) = u^{\frac{1}{9}}$.

Proof. Let $\epsilon > 0$, let $\sigma_t(s)$ be the parabolic approximation of γ at t, and let $\eta_t(s)$ be the osculating circle parametrized with respect to arclength at $\gamma(t)$. (It will be a line in the case that $\kappa_\gamma^{\text{abs}}(t) = 0$.) Abbreviate σ_t and η_t as σ and η, respectively. Then

$$\gamma(t) = \sigma(0) = \eta(0), \quad \gamma'(t) = \sigma'(0) = \eta'(0), \quad \text{and} \quad \gamma''(t) = \sigma''(0) = \eta''(0),$$

so σ is also a parabolic approximation of η at $\gamma(t)$. Now observe the following, omitting all input variables:

$$\begin{aligned}
\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[t-\delta,t+\delta]})) - 2\pi\right| &= \Big|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[t-\delta,t+\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]})) \\
&\qquad + \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\eta|_{[-\delta,\delta]})) \\
&\qquad + \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\eta|_{[-\delta,\delta]})) - 2\pi\Big| \\
&\leq \left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[t-\delta,t+\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]}))\right| \\
&\quad + \left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\sigma|_{[-\delta,\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\eta|_{[-\delta,\delta]}))\right| \\
&\quad + \left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\eta|_{[-\delta,\delta]})) - 2\pi\right|
\end{aligned}$$

By Lemma 3.12 there exists $d_1, d_2 > 0$ such that the first and second terms are less than $\frac{\epsilon}{3}$ for $0 < u < d_1$ and $0 < u < d_2$, respectively, both for all t. By Lemma 3.10 there exists $d_3 > 0$ such that the third term is less than $\frac{\epsilon}{3}$ for $0 < u < d_3$ for all t. By defining $d = \min\{d_1, d_2, d_3\}$, we then have that

$$\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda}(\gamma|_{[-\delta,\delta]})) - 2\pi\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever u such that $0 < u < d$ for all t, which is the desired result. $\square$

Next we show that the total absolute curvature of the image of almost all of our curve under inversion is approximated by the total absolute curvature of a broken inverted curve.

Lemma 3.14. *Let $\gamma : [0,\ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. For all $\epsilon > 0$ there exists $d > 0$ such that for all $t \in [0,\ell]$ and $\vec{\omega}$ satisfying $|\vec{\omega}| = 1$ and $\vec{\omega} \perp \gamma'(t)$ one has*

$$\left|\kappa_{tot}^{abs}(I_{r,\lambda(0)}(\gamma|_{[t+\delta(u),\ell+t-\delta(u)]})) - \kappa_{tot}^{abs}(I_{r,\lambda(u)}(\gamma|_{[t+\delta(u),\ell+t-\delta(u)]}))\right| < \epsilon$$

whenever $0 < u < d$, where $\lambda(u) = \gamma(t) + u\vec{\omega}$ and $\delta(u) = u^{\frac{1}{9}}$.

Proof. Let $\epsilon > 0$. We will first prove the case where $t = 0$, then show that the proof for $t = 0$ can be extended to work for any t. Let $\Delta_\gamma(s,u) = |\gamma(s) - \lambda(u)|^2$ and use the abbreviations $\Delta_0 = \Delta_\gamma(s,0)$ and $\Delta_u = \Delta_\gamma(s,u)$. We begin with the following, omitting all input variables and defining N_i and D as shown below:

$$\begin{aligned}
&\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma|_{[\delta,\ell-\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[\delta,\ell-\delta]}))\right| \\
&= \left| \int_\delta^{\ell-\delta} \left|\gamma'' - \gamma'\frac{\Delta_0'}{\Delta_0} - (\gamma - \lambda(0))\frac{\Delta_0''}{\Delta_0} + (\gamma - \lambda(0))\frac{(\Delta_0')^2}{\Delta_0^2}\right| ds \right. \\
&\qquad \left. - \int_\delta^{\ell-\delta} \left|\gamma'' - \gamma'\frac{\Delta_u'}{\Delta_u} - (\gamma - \lambda(u))\frac{\Delta_u''}{\Delta_u} + (\gamma - \lambda(u))\frac{(\Delta_u')^2}{\Delta_u^2}\right| ds \right| \\
&\leq \int_\delta^{\ell-\delta} |\underbrace{\gamma'' - \gamma''}_{N_1}|\, ds \\
&\quad + \int_\delta^{\ell-\delta} \Bigg[\Bigg| -\underbrace{\left(\gamma'\Delta_0'\Delta_0\Delta_u^2 - \gamma'\Delta_u'\Delta_u\Delta_0^2\right)}_{N_2} \\
&\qquad - \underbrace{\left((\gamma - \lambda(0))\Delta_0''\Delta_0\Delta_u^2 - (\gamma - \lambda(u))\Delta_u''\Delta_u\Delta_0^2\right)}_{N_3} \\
&\qquad + \underbrace{\left((\gamma - \lambda(0))(\Delta_0')^2\Delta_u^2 - (\gamma - \lambda(u))(\Delta_u')^2\Delta_0^2\right)}_{N_4} \Bigg| \Big/ \underbrace{(\Delta_0^2\Delta_u^2)}_{D} \Bigg] ds
\end{aligned} \tag{3}$$

Observe that these integrals remain the same if they are integrated over $[-\frac{\ell}{2}, -\delta] \cup [\delta, \frac{\ell}{2}]$. Obviously we have that

$$N_1 = \gamma'' - \gamma'' = 0.$$

We shall use a representation of γ in terms of its parabolic approximation at t (recall that we assumed without loss of generality $t = 0$.) That is,

$$\gamma(s) = A + Bs + Cs^2 + L$$

for all $s \in [-\frac{\ell}{2}, \frac{\ell}{2}]$, where $A = \gamma(0)$, $B = \gamma'(0)$, and $C = \frac{\gamma(0)}{2}$. Note that L will be the same L given in Lemma 3.11. We know from Lemma 3.11 that $\frac{L}{s^2}$ and $\frac{L'}{s}$ are continuously extendable to $[-\frac{\ell}{2}, \frac{\ell}{2}]$. Call these extensions $\tilde{L}$ and $\hat{L}$, respectively, and note that then $s^2\tilde{L} = L$ and $s\hat{L} = L'$. We can then write γ and its derivatives as follows:

$$\begin{aligned} \gamma(s) - \lambda(0) &= sB + s^2C + s^2\tilde{L} & \gamma'(s) &= B + s2C + s\hat{L} \\ \gamma(s) - \lambda(u) &= sB + s^2C + s^2\tilde{L} - u\vec{\omega} & \gamma''(s) &= 2C + L'' \end{aligned}$$

Now recall that

$$B \bullet C = \gamma'(0) \bullet \gamma''(0) = 0 \quad \text{and} \quad B \bullet B = \gamma'(0) \bullet \gamma'(0) = 1,$$

Since γ was parametrized by arclength, $B \bullet C = 0$ and $B \bullet B = 1$. In addition, our choice of ω ensures $B \bullet \vec{\omega} = 0$ and $\vec{\omega} \bullet \vec{\omega} = 1$. Using this in conjunction with the above expansions allows us to make the following calculations, as in Lemma 3.12:

$$\begin{aligned} \Delta_u = {} & s^2 + u^2 + s^3(2B \bullet \tilde{L}) + s^4(C \bullet C + 2C \bullet \tilde{L} + \tilde{L} \bullet \tilde{L}) \\ & - s^2u(2C \bullet \vec{\omega} + 2\tilde{L} \bullet \vec{\omega}) \end{aligned} \tag{4}$$

$$\begin{aligned} \Delta'_u = {} & 2s + s^2(2B \bullet \hat{L} + 2B \bullet \tilde{L}) + s^3(4C \bullet C + 2C \bullet \hat{L} + 4C \bullet \tilde{L} + 2\tilde{L} \bullet \hat{L}) \\ & - su(4C \bullet \vec{\omega} + 2\hat{L} \bullet \vec{\omega}) \end{aligned} \tag{5}$$

$$\begin{aligned} \Delta''_u = {} & 2 + s(2B \bullet L'' + 4B \bullet \hat{L}) \\ & + s^2(12C \bullet C + 4C \bullet \tilde{L} + 8C \bullet \hat{L} + 2C \bullet L'' + 2\tilde{L} \bullet L'' + 2\hat{L} \bullet \hat{L}) \\ & - u(4C \bullet \vec{\omega} + 2L'' \bullet \vec{\omega}) \end{aligned} \tag{6}$$

We obtain Δ_0, Δ'_0, and Δ''_0 by setting $u = 0$ in the above equations:

$$\Delta_0 = s^2 + s^3(2B \bullet \tilde{L}) + s^4(C \bullet C + 2C \bullet \tilde{L} + \tilde{L} \bullet \tilde{L}) \tag{7}$$

$$\Delta'_0 = 2s + s^2(2B \bullet \hat{L} + 2B \bullet \tilde{L}) + s^3(4C \bullet C + 2C \bullet \hat{L} + 4C \bullet \tilde{L} + 2\tilde{L} \bullet \hat{L}) \tag{8}$$

$$\begin{aligned} \Delta''_0 = {} & 2 + s(2B \bullet L'' + 4B \bullet \hat{L}) \\ & + s^2(12C \bullet C + 4C \bullet \tilde{L} + 8C \bullet \hat{L} + 2C \bullet L'' + 2\tilde{L} \bullet L'' + 2\hat{L} \bullet \hat{L}) \end{aligned} \tag{9}$$

In order to simplify our calculations, we will use the following consolidated versions of each of the Δ-terms (Expressions (4) to (9)), with X_i being defined appropriately. Note that although the X_i will not be constant, they are all bounded in s and u for $t=0$. We then write the following:

$$\begin{array}{ccc} \Delta_u = X_1 + uX_2 & \Delta'_u = X_3 + uX_4 & \Delta''_u = X_5 + uX_6 \\ \Delta_0 = X_1 & \Delta'_0 = X_3 & \Delta''_0 = X_5 \end{array}$$

We now calculate the cross terms of Expression (3). We define T_i as we proceed to further consolidate expressions. The T_i are all be bounded in s and t for $t=0$. First, the cross coefficients:

$$\begin{array}{lll} \Delta'_0\Delta_0\Delta_u^2 = X_1^3X_3 + uT_1 & \Delta''_0\Delta_0\Delta_u^2 = X_1^3X_5 + uT_3 & (\Delta'_0)^2\Delta_u^2 = X_1^2X_3^2 + uT_5 \\ \Delta'_u\Delta_u\Delta_0^2 = X_1^3X_3 + uT_2 & \Delta''_u\Delta_u\Delta_0^2 = X_1^3X_5 + uT_4 & (\Delta'_u)^2\Delta_0^2 = X_1^2X_3^2 + uT_6 \end{array}$$

Recall $A = \gamma(0)$, so $\lambda(u) = A + u\vec{\omega}$. We can then calculate the following:

$$\begin{aligned} N_2 &= \gamma'(X_1^3X_3 + uT_1) - \gamma'(X_1^3X_3 + uT_2) = uT_7 \\ N_3 &= (\gamma - A)(X_1^3X_5 + uT_3) - (\gamma - A - u\vec{\omega})(X_1^3X_5 + uT_4) = uT_8 \\ N_4 &= (\gamma - A)(X_1^2X_3^2 + uT_5) - (\gamma - A - u\vec{\omega})(X_1^2X_3^2 + uT_6) = uT_9 \end{aligned}$$

This allows us to write $-N_2 - N_3 + N_4 = uT_{10}$. Since T_{10} is a sum and product of terms bounded in s and u for $t=0$, we know that there exists some $M \in \mathbb{R}^+$ such that $M \geq |T_{10}|$ for all $s \in [0,\ell]$ and $u \in [0,r]$ when $t=0$, where r is as given in Proposition 3.3. Hence, we have that

$$|-N_2 - N_3 + N_4| \leq uM.$$

We now address the last term in Expression (3), D.

Claim 3.14.1. *Let κ_m be the maximum curvature of γ, which we know exists, since γ is C^2 and has a compact domain. If $|s| \leq \frac{1}{\kappa_m}$, we have that*

$$|D| \geq \frac{u^{\frac{8}{9}}}{256}.$$

Proof of Claim 3.14.1: As γ is closed, recall that $\gamma(\ell - s) = \gamma(-s)$. (See Figure 5.) We will split $|s| \leq \frac{1}{\kappa_m}$ into the cases $\frac{-1}{\kappa_m} \leq s \leq 0$ and $0 \leq s \leq \frac{1}{\kappa_m}$. The proofs are similar, so we show only the latter case.

Let $|s| \leq \frac{1}{\kappa_m}$, and define $P = (\gamma'(0))^\perp$, the normal hyperplane at $\gamma(0)$. Recall that γ is parametrized by arclength and define $f : \left[\frac{-1}{\kappa_m}, \frac{1}{\kappa_m}\right] \to \mathbb{R}$ by $f(s) = (\gamma(s) - \gamma(0)) \cdot \gamma'(0)$. Observe that

$$\begin{aligned}
\operatorname{dist}(\gamma(s), P) &= \left|\operatorname{proj}_{\gamma'(0)}(\gamma(s) - \gamma(0))\right| \\
&= \left|\frac{(\gamma(s) - \gamma(0)) \cdot \gamma'(0)}{\gamma'(0) \cdot \gamma'(0)} \gamma'(0)\right| \\
&= |(\gamma(s) - \gamma(0)) \cdot \gamma'(0)| = |f(s)|.
\end{aligned}$$

We then have $f'(s) = \gamma'(s) \cdot \gamma'(0)$ and $f''(s) = \gamma''(s) \cdot \gamma'(0)$. In addition, note $f(0) = 0$ and $f'(0) = 1$. The definition of f and Cauchy-Schwarz imply

$$|f''(s)| = |\gamma''(s) \cdot \gamma'(0)| \leq |\gamma''(s)||\gamma'(0)| = |\gamma''(s)| \leq \kappa_m \implies f''(s) \geq -\kappa_m.$$

Integrating this inequality and using $f'(0) = 1$ then gives $f'(s) \geq 1 - \kappa_m s$. We then integrate again and use $f(0) = 0$ to obtain

$$f(s) \geq s - \frac{\kappa_m}{2} s^2. \tag{10}$$

For $0 \leq s \leq \frac{1}{\kappa_m}$ we have

$$\frac{1}{\kappa_m} \geq s \implies \frac{1}{2} \geq \frac{\kappa_m}{2} s \implies 1 - \frac{\kappa_m}{2} s \geq \frac{1}{2} \implies s - \frac{\kappa_m}{2} s^2 \geq \frac{s}{2}.$$

Substitution of this inequality into Inequality (10) implies $f(s) \geq \frac{s}{2}$ for $0 \leq s \leq \frac{1}{\kappa_m}$. With a similar proof, one can conclude that $|f(s)| \geq \frac{|s|}{2}$ for $\frac{-1}{\kappa_m} \leq s \leq \frac{1}{\kappa_m}$. As $|f(s)| = \operatorname{dist}(\gamma(s), P)$ and $\lambda(0), \lambda(u) \in P$, we may then conclude

$$|\gamma(s) - \lambda(0)| \geq \frac{|s|}{2} \quad \text{and} \quad |\gamma(s) - \lambda(u)| \geq \frac{|s|}{2}.$$

Recall the definition of D, which allows us to write

$$|D| = \Delta_0^2 \Delta_u^2 = |\gamma(s) - \lambda(0)|^4 |\gamma(s) - \lambda(u)|^4 \geq \left(\frac{s}{2}\right)^4 \left(\frac{s}{2}\right)^4 = \frac{s^8}{256}.$$

As $|s| \geq \delta(u) = u^{\frac{1}{9}}$, we may conclude that $|D| \geq \frac{u^{\frac{8}{9}}}{256}$, as desired. ■

We shall use Claim 3.14.1 to prove a second claim.

Claim 3.14.2. *There exists sufficiently small u so that*

$$|D| \geq \frac{u^{\frac{8}{9}}}{256}.$$

Proof of Claim 3.14.2: We have shown this for $|s| \leq \frac{1}{\kappa_m}$ in Claim 3.14.1. As such, we may assume that $|s| \geq \frac{1}{\kappa_m}$. Since γ is compact and C^2 and $|s| \geq \frac{1}{\kappa_m}$, there exists some $K_1 \in \mathbb{R}^+$ such that $|\gamma(s)-\lambda(0)| = |\gamma(s)-\gamma(0)| \geq K_1$. Now choose u small enough so that $\lambda(u)$ is contained the normal neighborhood of γ guaranteed to exist by Proposition 3.3. Again, since γ is compact and C^2 and $|s| \geq \frac{1}{\kappa_m}$, there exists some $K_2 \in \mathbb{R}^+$ such that $|\gamma(s) - \lambda(u)| \geq K_2$. We can then use the definition of D to write

$$|D| = \Delta_0^2 \Delta_u^2 = |\gamma(s) - \lambda(0)|^4 |\gamma(s) - \lambda(u)|^4 \geq K_1^4 K_2^4.$$

Hence, we simply choose u to be small enough so that $K_1^4 K_2^4 \geq \frac{u^{\frac{8}{9}}}{256}$ to obtain the desired result. ■

Having proved Claim 3.14.2, we now have for sufficiently small u that

$$\int\limits_{[-\frac{\ell}{2},-\delta]\cup[\delta,\frac{\ell}{2}]} |N_1| + \frac{|-N_2 - N_3 + N_4|}{|D|}\, ds \leq \int\limits_{[-\frac{\ell}{2},-\delta]\cup[\delta,\frac{\ell}{2}]} \frac{256uM}{u^{\frac{8}{9}}} \leq 256\ell M u^{\frac{1}{9}}.$$

It then follows that there exists some $d > 0$ such that $256\ell M u^{\frac{1}{9}} < \epsilon$ whenever $0 < u < d$ for $t = 0$. Thus, we have that

$$\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma|_{[\delta,\ell+\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[\delta,\ell+\delta]}))\right| < \epsilon$$

whenever $0 < u < d$ for $t = 0$, as desired. This completes the proof for $t = 0$. This can be extended to all t by the same method described at the conclusion of the proof of Lemma 3.12, as well as choosing K_1, K_2 uniformly for all t by using continuity and compactness. □

Lemmas 3.13 and 3.14 combine to prove Proposition 3.15.

Proposition 3.15. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. For all $\epsilon > 0$ there exists $d > 0$ such that for all $t \in [0, \ell]$ and $\vec{\omega}$ satisfying $|\vec{\omega}| = 1$ and $\vec{\omega} \perp \gamma'(t)$ one has*

$$\left|\kappa_{tot}^{abs}(I_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{tot}^{abs}(I_{r,\lambda(u)}(\gamma))\right| < \epsilon,$$

whenever $0 < u < d$, where $\lambda(u) = \gamma(t) + u\vec{\omega}$.

Proof. Let $\epsilon > 0$ and define $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ by $\delta(u) = u^{\frac{1}{9}}$. Then

$$\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma)) = \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t+\delta,\ell+t-\delta]})) + \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t-\delta,t+\delta]})).$$

Now observe the following:

$$\begin{aligned} &|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma))| \\ \leq\ &|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t+\delta,\ell+t-\delta]}))| + |2\pi - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t-\delta,t+\delta]}))| \\ \leq\ &|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma|_{[t+\delta,\ell+t-\delta]}))| \\ &+ |\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma|_{[t+\delta,\ell+t-\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t+\delta,\ell+t-\delta]}))| \qquad (11) \\ &+ |\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t-\delta,t+\delta]})) - 2\pi| \end{aligned}$$

Lemma 3.7 tells us that the integrand of $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma))$ will be bounded, so there exists some $d_1 > 0$ such that

$$|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma|_{[t+\delta,\ell+t-\delta]}))| < \frac{\epsilon}{3}$$

when $0 < u < d_1$ for all t (since $\delta = u^{\frac{1}{9}}$). By Lemma 3.14 there exists $d_2 > 0$ such that

$$|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma|_{[t+\delta,\ell+t-\delta]})) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t+\delta,\ell+t-\delta]}))| < \frac{\epsilon}{3}$$

when $0 < u < d_2$ for all t. By Lemma 3.13 there exists $d_3 > 0$ such that

$$|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma|_{[t-\delta,t+\delta]})) - 2\pi| < \frac{\epsilon}{3}$$

when $0 < u < d_3$ for all t. Set $d = \min\{d_1, d_2, d_3\}$. Then the substitution of these three inequalities into Expression (11) yields

$$|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma))| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

when $0 < u < d$ for all t, which is the desired result. $\square$

Corollary 3.16 provides a bound for the total absolute curvature of an inverted image of a curve for inversion centers sufficiently close to the curve.

Corollary 3.16. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength, and define*

$$M_1 = \max_{t \in [0,\ell]} \kappa_{tot}^{abs}(I_{r,\gamma(t)}(\gamma)).$$

Then for every $\epsilon > 0$ there exists some $d_4 \in \mathbb{R}^+$ such that for all $r \in \mathbb{R}^+$ and all $c \in \mathbb{R}^n$ such that $dist(c, \gamma) < d_4$,

$$0 \leq \kappa_{tot}^{abs}(I_{r,c}(\gamma)) \leq M_1 + 2\pi + \epsilon.$$

Remark. Notice immediately that $0 \leq \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma))$, as the total absolute curvature cannot be negative. Note that the lower bound of 0 is sharp, as when $c \in \gamma$, $I_{r,c}(\gamma)$ may not be closed. (In fact, $I_{r,c}(\gamma)$ is a line if γ is a circle and $c \in \gamma$.) If $c \notin \gamma$, then we know that the greatest lower bound of $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma))$ will be 2π, as $\mathrm{I}_{r,c}(\gamma)$ will be a closed curve [4],[1].

Proof. Recall from Part (ii) of Corollary 3.1 that $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma))$ will not depend on r. Let $\epsilon > 0$. Note that M_1 as defined above is guaranteed to exist by the fact that $[0, \ell]$ is compact and the integrand giving $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\gamma(t)}(\gamma))$ is bounded on $[-\frac{\ell}{2}, \frac{\ell}{2}] \times [0, \ell]$ by Lemma 3.7. We have two cases.

Case 1: $c \in \gamma$. If $c \in \gamma$, then we immediately have that $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma)) \leq M_1$.

Case 2: $c \notin \gamma$. Since γ is a C^2 curve with a compact domain, Proposition 3.3 tells us that there exists some $d_1 > 0$ such that Parts (i) to (iii) of Proposition 3.3 are satisfied when $0 < \text{dist}(c, \gamma) < d_1$. Since d_1 is being chosen so that Proposition 3.3 holds, we can take $\lambda(u) = \gamma(t) + u\vec{\omega}$, where $|\vec{\omega}| = 1$, $\vec{\omega} \perp \gamma'(t)$, $\lambda(0) = \gamma(t)$, and $\lambda(u_c) = c$, where $u_c = \text{dist}(c, \gamma)$. It also follows from Proposition 3.15 that there exists some $d > 0$ such that

$$\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma))\right| < \epsilon$$

when $0 < u < d$ for all t. Define $d_4 = \min\{d_1, d\}$. Let c be such that $0 < \text{dist}(c, \gamma) < d_4$. Then for all t:

$$\begin{aligned} &\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u_c)}(\gamma))\right| < \epsilon \\ \implies &\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma)) < \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\gamma(t)}(\gamma)) + 2\pi + \epsilon \end{aligned}$$

It then follows from the definition of M_1 that $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma)) \leq M_1 + 2\pi + \epsilon$ when $0 < \text{dist}(c, \gamma) < d$, as desired. □

3.4 Proofs of main results

We first obtain the main result (Theorem 3.19) for C^2 simple closed curves, then generalize to C^2 nonsimple closed curves as Theorem 3.20. Of course the latter implies the former, but the former's proof is such that it is considerably easier to prove the latter from the former than to write a single general proof.

3.4.1 Simple curves

The bound developed in Corollary 3.16 can then be united with the bound for inversion center in a neighborhood of infinity (Proposition 3.2) to obtain a bound for inversion centers anywhere in $\mathbb{R}^n$.

Proposition 3.17. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. Then there exists some $M_0 \in \mathbb{R}^+$ such that for all r, c we have $\kappa_{tot}^{abs}(I_{r,c}(\gamma)) \leq M_0$.*

Proof. Recall from Proposition 3.2 that $\lim_{|c|\to\infty} \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma)) = \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\gamma)$ for any $r \in \mathbb{R}^+$. Hence, we have that there exists some $R \in \mathbb{R}^+$ such that

$$\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma)) \leq \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\gamma) + 1$$

for all $c \in X$, where $X = \{c \in \mathbb{R}^n \mid |c| > R\}$. Corollary 3.16 tells us that there exists some $d > 0$ and M_1 such that

$$\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma)) \leq M_1$$

when $c \in N$, where $Y = \{c \in \mathbb{R}^n \mid \mathrm{dist}(c, \gamma) < d\}$. We know from Part (iii) of Corollary 3.1 that $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma))$ will be continuous on the compact set $Z = \mathbb{R}^n - (X \cup Y)$. Hence, there exists some M_2 such that

$$\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma)) \leq M_2$$

when $c \in Z$. Set $M_0 = \max\{\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\gamma) + 1, M_1, M_2\}$ to complete the proof. □

Proposition 3.18. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength. Then there exists some $M_0 \in \mathbb{R}^+$ such that $\kappa_{tot}^{abs}(T(\gamma)) \leq M_0$ for all Möbius transformations T.*

Proof. For any affine similarity A, we can write $A(x) = aBx + y$, where $a \in \mathbb{R}^+$, $y \in \mathbb{R}^n$, and B is an orthogonal linear transformation of $\mathbb{R}^n$. It is then straightforward to show that $\mathrm{I}_{ar,A(c)}(A(x)) = A(\mathrm{I}_{r,c}(x))$.

Recall that any Möbius transformation T can be decomposed into a sequence of at most one each of translations, rotations, reflections, dilations, and inversions. We then write T as $T = A_2 \circ J_1 \circ A_1 = A_3 \circ J_2$, where the A_i are (possibly trivial) affine similarities and the J_i are either an inversion or an identity. Since $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\gamma)$ is invariant under affine transformations of γ,

$$\begin{aligned}&\{\kappa_{\mathrm{tot}}^{\mathrm{abs}}(T(\gamma)) \mid T \text{ is a Möbius Transformation}\} \\ &\qquad = \{\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma)) \mid r \in \mathbb{R}^+, c \in \mathbb{R}^n\} \cup \{\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\gamma)\}.\end{aligned}$$

By Proposition 3.17 there exists M_0 with $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma)) \leq M_0$ for all $r \in \mathbb{R}^+$ and $c \in \mathbb{R}^n$. Since $\lim_{|c|\to\infty} \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma)) = \kappa_{\text{tot}}^{\text{abs}}(\gamma)$, we then also have that $M_0 \geq \kappa_{\text{tot}}^{\text{abs}}(\gamma)$. Hence, $\kappa_{\text{tot}}^{\text{abs}}(T(\gamma)) \leq M_0$ for any Möbius transformation T. $\square$

Theorem 3.19 tells us that the total absolute curvature of the inverted image of a curve as a function of inversion center is removably discontinuous.

Theorem 3.19. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a C^2 simple closed curve parametrized by arclength, and define the map $\hat{\Phi} : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ by*

$$\hat{\Phi}(r, c) = \begin{cases} \kappa_{tot}^{abs}(I_{r,c}(\gamma)) + 2\pi & c \in \gamma([0, \ell]) \\ \kappa_{tot}^{abs}(I_{r,c}(\gamma)) & c \notin \gamma([0, \ell]) \end{cases}.$$

Then $\hat{\Phi}$ is continuous and bounded. In fact,

$$\lim_{|c|\to\infty} \hat{\Phi}(r, c) = \kappa_{tot}^{abs}(\gamma) \quad \text{for all } r \in \mathbb{R}^+.$$

Proof. $\hat{\Phi}$ is bounded by Proposition 3.17 and

$$\lim_{|c|\to\infty} \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma) = \kappa_{\text{tot}}^{\text{abs}}(\gamma).$$

for all r is the result of Proposition 3.2. $\hat{\Phi}$ is independent of r and $\hat{\Phi}(r, c)$ is continuous at every $(r, c) \in \mathbb{R}^+ \times (\mathbb{R}^n - \gamma)$ by Parts (ii) and (iii) of Corollary 3.1, respectively. It only remains to show that $\hat{\Phi}$ is continuous at each $c_0 \in \gamma$. Let $\epsilon > 0$ and t_0 be such that $c_0 = \gamma(t_0)$. Choose $d_1 > 0$ so that it satisfies the following conditions, (i)-(iii):

(i) $|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\gamma(t_0)}(\gamma)) - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\gamma(t)}(\gamma))| < \frac{\epsilon}{2}$ for $|t_0 - t| < d_1$, possible by Proposition 3.9.

(ii) Proposition 3.3 is satisfied for d_1.

We take $\lambda(u) = \gamma(t) + u\vec{\omega}$ for each given t and each $\vec{\omega} \perp \gamma'(t)$ with $|\vec{\omega}| = 1$.

(iii) $\left|\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,\lambda(u)}(\gamma))\right| < \frac{\epsilon}{2}$ when $0 < u < d_1$ for all t and $\omega \perp \gamma'(t)$ with $|\omega| = 1$, which is possible via Proposition 3.15 and Condition (ii).

Define the open set W by

$$W = \{\gamma(t) + u\vec{\omega} \mid t \in (t_0 - d_1, t_0 + d_1), \vec{\omega} \perp \gamma'(t), |\vec{\omega}| = 1, 0 \leq u < d_1\}.$$

Since W is open, there exists $d \in \mathbb{R}^+$ such that $0 < d < d_1$ and $B_d^n(c_0) \subseteq W$.

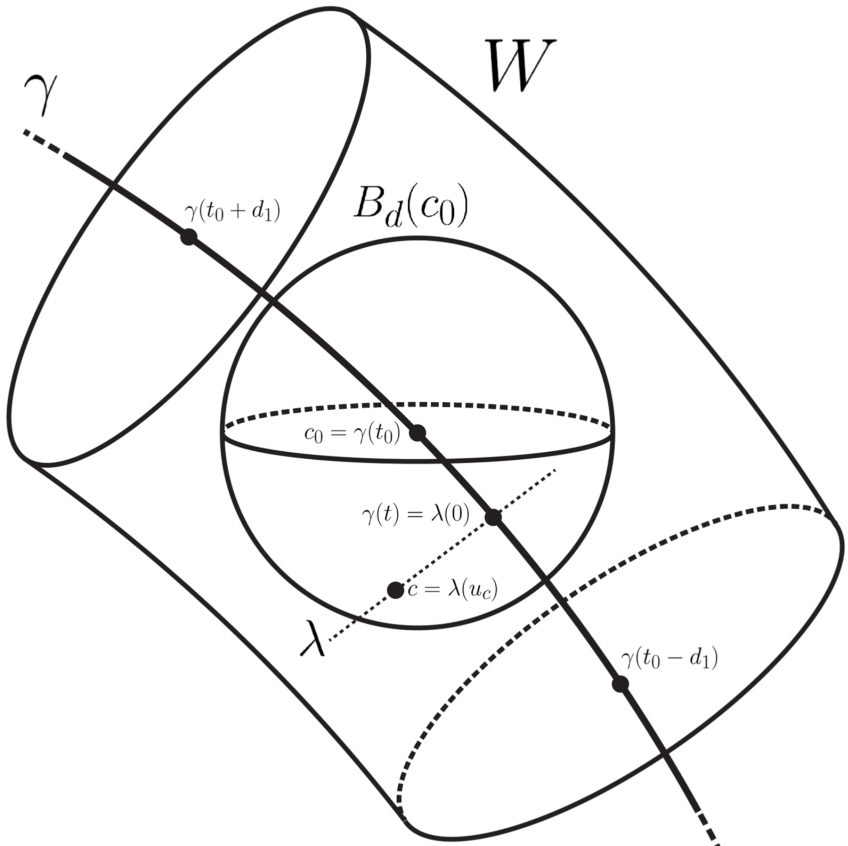

Figure 6: An illustration of the construction of W and $B^n_d(c_0)$. Note that the particular c shown in the diagram corresponds with case 2.

Take an arbitrary $c \in B^n_d(c_0) \subseteq W$. The definition of W and Part (i) of Proposition 3.3 tell us that if $t \in [0, \ell]$ is such that $|c - \gamma(t)| = \mathrm{dist}(c, \gamma)$, then $t \in (t_0 - d_1, t_0 + d_1)$. Consequently, $c = \gamma(t) + u\vec{\omega} = \lambda(u_c)$ for some $t \in (t_0 - d_1, t_0 + d_1)$ and $\vec{\omega} \perp \gamma'(t)$ with $|\vec{\omega}| = 1$, and

$$0 \leq u = \mathrm{dist}(c, \gamma) \leq \mathrm{dist}(c, c_0) < d < d_1.$$

We claim $|\hat{\Phi}(r, c_0) - \hat{\Phi}(r, c)| < \epsilon$ when $|c_0 - c| < d$. There are two cases.

Case 1: $c \in \gamma$. The definition of $\hat{\Phi}$ and Condition (i) tell us that

$$\begin{aligned}|\hat{\Phi}(r, c_0) - \hat{\Phi}(r, c)| &= |(\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\gamma(t_0)}(\gamma)) + 2\pi) - (\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\gamma(t)}(\gamma)) + 2\pi)| \\ &= |\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\gamma(t_0)}(\gamma)) - \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\gamma(t)}(\gamma))| < \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Case 2: $c \notin \gamma$. Note that

$$|\hat{\Phi}(r, c_0) - \hat{\Phi}(r, c)| = |\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c_0}(\gamma)) + 2\pi - \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma))|.$$

Condition (ii) and our definition of λ then allow us to write $c_0 = \gamma(t_0)$, $\gamma(t) = \lambda(0)$, $u_c = \mathrm{dist}(c, \gamma)$ and $c = \lambda(u_c)$ when $0 < u_c < d$, since $c \notin \gamma$. By combining Conditions (i) and (iii) and Proposition 3.15, we have

$$\begin{aligned}&|\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c_0}(\gamma)) + 2\pi - \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma))| \\ &\leq |\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\gamma(t_0)}(\gamma)) - \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\gamma(t)}(\gamma))| \\ &\quad + |\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\lambda(0)}(\gamma)) + 2\pi - \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,\lambda(u_c)}(\gamma))| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, we have $\hat{\Phi}$ is continuous at every $c_0 \in \gamma$, so $\hat{\Phi}$ is continuous. $\square$

Remark. Since $\lim_{|c|\to\infty} \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\gamma)) = \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\gamma)$, we also know that $\hat{\Phi}$ is uniformly continuous.

3.4.2 Non-simple curves

We can generalize Theorem 3.19 to non-simple curves in a natural way.

Theorem 3.20. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a regular C^2 closed curve parametrized with respect to arclength, possess finitely many points of self-intersection, and be such that $\gamma^{-1}(c)$ is a finite set for all $c \in \mathbb{R}^n$. Then*

$$\hat{\Psi}(r, c) = \kappa_{tot}^{abs}(I_{r,c}(\gamma)) + 2\pi m(c)$$

is continuous, where $m(c) = card(\gamma^{-1}(c))$; that is, $m(c)$ is the multiplicity of γ at c when $c \in \gamma$ and $m(c) = 0$ if $c \notin \gamma$.

Proof. We begin with the simplest case; specifically, assume γ has one self-intersection point p of multiplicity two. Let $n \geq 3$, as we will prove $n = 2$ later. If $c \notin \gamma$, then we have $\hat{\Psi}$ continuous by Part (iii) of Corollary 3.1, so assume that $c \in \gamma$. Then there exists some sufficiently small, closed ball B of radius $\epsilon > 0$ at p such that B separates γ into three parts: two disjoint arcs β_1 and β_2 outside B and the self-intersection contained in B, which satisfies $\gamma^{-1}(\gamma \cap B) = [t_1, t_2] \cup [t_3, t_4]$ where $t_1 < t_2 < t_3 < t_4$. We shall show continuity for $c \in B$ and $c \notin B$ separately.

Case 1: $c \in B$. Choose points q_1 and q_2 on β_1 and β_2, respectively; then q_1 and q_2 separate γ into two open arcs α_1 and α_2. Construct a C^2 simple closed curve $A_1 = \alpha_1 \cup \theta_1$ by using an arc θ_1 disjoint from B. Construct A_2 similarly. Theorem 3.19 tells us that $\hat{\Phi}_{A_1} + \hat{\Phi}_{A_2}$ is continuous on $\mathbb{R}^n$, in particular, on B. As θ_1 and θ_2 are disjoint from B, we may also conclude that $\hat{\Phi}_{\theta_1} + \hat{\Phi}_{\theta_2}$ is continuous on B by Part (iii) of Corollary 3.1. Notice that

$$\hat{\Phi}_{A_1} + \hat{\Phi}_{A_2} - \hat{\Psi} = \hat{\Phi}_{\theta_1} + \hat{\Phi}_{\theta_2},$$

which allows us to conclude that $\hat{\Psi}$ is continuous on B.

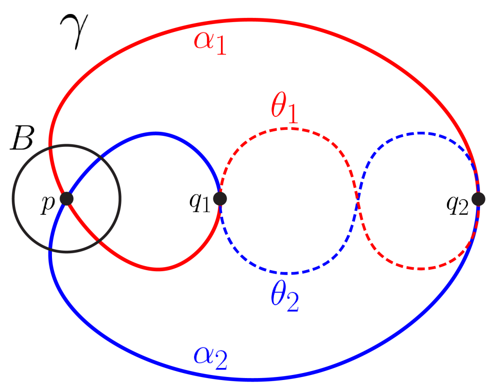

Figure 7: An illustration of the construction of θ_1 and θ_2 in case 1.

Case 2: $c \notin B$. Let D be a closed ball of radius ϵ' at p such that $0 < \epsilon' < \epsilon$ and D satisfies the same separation conditions as B. Choose two distinct points $p_1, p_2 \in \alpha_1 \cap \operatorname{int}(D)$ such that p is between p_1 and p_2. Let ξ_1 be the segment of α_1 between p_1 and p_2 in D, and let $\gamma_0 = \gamma - \xi_1$. Since $n \geq 3$, there exists a C^2 curve ξ_2 between p_1 and p_2 in D such that $\gamma_2 = \gamma_0 \cup \xi_2$ is a C^2 simple closed curve. Then $\hat{\Phi}_{\gamma_2}$ is continuous on $\mathbb{R}^n$ by Theorem 3.19. Further, both $\hat{\Phi}_{\xi_1}$ and $\hat{\Phi}_{\xi_2}$ are continuous on $\mathbb{R}^n - B$. Notice that

$$\hat{\Psi} = \hat{\Phi}_{\gamma_2} - \hat{\Phi}_{\xi_2} + \hat{\Phi}_{\xi_1},$$

which allows us to conclude that $\hat{\Psi}$ is continuous on $\mathbb{R}^n - B$. This concludes the proof for $n \geq 3$. Now say $n = 2$. Embed γ into $\mathbb{R}^3$ via the standard isometric inclusion $\mathbb{R}^2 \subseteq \mathbb{R}^3$; the above work tells us that $\hat{\Psi}$ is continuous on $\mathbb{R}^3$. As every inversion of $\mathbb{R}^2$ is a restriction of an inversion in $\mathbb{R}^3$, this implies continuity for all $c \in \mathbb{R}^2$. Hence, we have the continuity $\hat{\Psi}$ in the case where γ one self-intersection of multiplicity two.

We now generalize to the case with more than one self-intersection. Let $p_1, \ldots, p_k \in \mathbb{R}^n$ be the set of all multiple points of γ. Choose $q_1, \ldots, q_\ell, q_{\ell+1} = q_1 \in \gamma$ away from the p_j such that each $\alpha_i = \gamma|_{[q_i, q_{i+1}]}$ is simple and contains exactly one p_j; it is helpful to notice that any p_i is on multiple α_i. Take $B_1, \ldots, B_k$ disjoint closed balls of radius $\epsilon > 0$ at $p_1, \ldots, p_k$, choosing ϵ small enough to ensure that no q_i is contained $B = \bigcup_{j=1}^k B_j$ and each B_j satisfies $\gamma^{-1}(\gamma \cap B_j) = [t_{j,1}, t_{j,2}] \cup [t_{j,3}, t_{j,4}]$ where $t_{j,1} < t_{j,2} < t_{j,3} < t_{j,4}$. Next, construct $A_i = \alpha_i \cup \theta_i$ to be a C^2 simple closed curve avoiding B, as above. Then both $\hat{\Phi}_{A_i}$ and $\hat{\Phi}_{\theta_i}$ are continuous on B for all i, so

$$\sum_{i=1}^{\ell} \hat{\Phi}_{A_i} - \hat{\Psi} = \sum_{i=1}^{\ell} \hat{\Phi}_{\theta_i}$$

implies the continuity of $\hat{\Psi}$ on B. The proof of continuity on $\mathbb{R}^n - B$ is similar to the proof for the simpler case shown above. □

3.5 Generalization to open or piecewise curves

3.5.1 Open or piecewise C^2 curves

Proposition 3.18 can be generalized to the case where γ is open or piecewise C^2, as described by Proposition 3.22. Unfortunately, we cannot do the same for Theorem 3.19, as $\hat{\Phi}$ will be disconintuous at endpoints and corners of γ.

Definition 3.1. A continuous curve $\gamma : [0, \ell] \to \mathbb{R}^n$ is a *piecewise C^2 curve* if there exists a set $\{t_0, \ldots, t_k\}$ such that the following hold:

(i) $0 = t_0 < t_1 < \cdots < t_k = \ell$.

(ii) $\gamma_i = \gamma|_{[t_{i-1}, t_i]} : [t_{i-1}, t_i] \to \mathbb{R}^n$ is C^2 on $[t_{i-1}, t_i]$. Notice that this requires $\kappa^{\text{abs}}_{\gamma_i}(t) \leq c_i < \infty$ for $t \in [t_{i-1}, t_i]$ and $c_i \in \mathbb{R}^+$.

(iii) $\angle(\gamma_i'(t_i), \gamma_{i+1}'(t_i))$ is well-defined for all i. Notice that if the curve is closed then $\angle(\gamma_k'(\ell), \gamma_1'(0))$ must also be well-defined.

Definition 3.2. For a piecewise C^2 curve γ with k segments, its *total absolute curvature* is given according to the formula

$$\kappa_{\text{tot}}^{\text{abs}}(\gamma) = \sum_{i=1}^{k} \kappa_{\text{tot}}^{\text{abs}}(\gamma_i) + \sum_{i=1}^{m} \angle(\gamma_i'(t_i), \gamma_{i+1}'(t_i)),$$

where $m = k$ and $\gamma_{k+1} = \gamma_1$ if γ is closed, and $m = k - 1$ otherwise.

To prove Proposition 3.22, we first generalize Proposition 3.17.

Proposition 3.21. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a piecewise C^2 curve parametrized by arclength. Then there exists some $M \in \mathbb{R}^+$ such that for all $r \in \mathbb{R}^+$ and $c \in \mathbb{R}^n$ we have $\kappa_{tot}^{abs}(I_{r,c}(\gamma)) < M$.*

Proof. Let $\gamma_1, \ldots, \gamma_k$ be the C^2 segments of γ. (If we have the special case in which γ is closed and only fails to be C^2 at one point t_0, split γ into two segments by selecting some point $t_1 \neq t_0$ to use as a second break point.)

Since γ is a piecewise C^2 curve, we know that each γ_i has bounded pointwise curvature on $[t_{i-1}, t_i]$, so we can construct some simple closed C^2 curve $\Gamma_i : [0, L_i] \to \mathbb{R}^n$ for each i such that $\Gamma_i(t) = \gamma_i(t)$ for $t \in [t_{i-1}, t_i]$. By Proposition 3.17 there exists M_i such that $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\Gamma_i)) \leq M_i$ for all $r \in \mathbb{R}^+$ and $c \in \mathbb{R}^n$. Hence, $\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma_i)) \leq M_i$ for all $r \in \mathbb{R}^+$ and $c \in \mathbb{R}^n$. As inversions are conformal transformations, we know for all $r \in \mathbb{R}^+$ and $c \in \mathbb{R}^n$ that

$$\angle(D\mathrm{I}_{r,c}(\gamma_i'(t_i)), D\mathrm{I}_{r,c}(\gamma_{i+1}'(t_i))) \leq \angle(\gamma_i'(t_i), \gamma_{i+1}'(t_i)),$$

with equality holding except when the inversion center c satisfies $c = \gamma(t_i)$. (In that case we take $\angle(D\mathrm{I}_{r,c}(\gamma_i'(t_i)), D\mathrm{I}_{r,c}(\gamma_{i+1}'(t_i))) = 0$, as the angle is eliminated when c goes to infinity.) Let $m = k$ and $\gamma_{k+1} = \gamma_1$ if γ is closed, and $m = k - 1$ otherwise. It then follows for all $r \in \mathbb{R}^+$ and $c \in \mathbb{R}^n$ that

$$\begin{aligned}\kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma)) &= \sum_{i=1}^{k} \kappa_{\text{tot}}^{\text{abs}}(\mathrm{I}_{r,c}(\gamma_i)) + \sum_{i=1}^{m} \angle(D\mathrm{I}_{r,c}(\gamma_i'(t_i)), D\mathrm{I}_{r,c}(\gamma_{i+1}'(t_i))) \\ &\leq \sum_{i=1}^{k} M_i + \sum_{i=1}^{m} \angle(\gamma_i'(t_i), \gamma_{i+1}'(t_i)).\end{aligned}$$

Hence, choosing

$$M = \sum_{i=1}^{k} M_i + \sum_{i=1}^{m} \angle(\gamma_i'(t_i), \gamma_{i+1}'(t_i))$$

completes the proof. □

Proposition 3.22 is proved similarly to Proposition 3.18.

Proposition 3.22. *Let $\gamma : [0, \ell] \to \mathbb{R}^n$ be a piecewise simple C^2 curve parametrized by arclength. Then there exists some $M_0 \in \mathbb{R}^+$ such that $\kappa_{tot}^{abs}(T(\gamma)) \leq M_0$ for all Möbius transformations T.*

Remark. Proposition 3.22 generalizes to all piecewise C^2 curves with finitely many self-intersections—simply partition $[0, \ell]$ so each segment of γ_i is simple.

3.5.2 Polygonal curves

We conclude with an elementary result for polygonal curves, Proposition 3.24. This proposition provided the motivation for Theorem 3.19 and demonstrated the implausibility of using linear approximations for the proof of the theorem.

Lemma 3.23. *Let $\lambda : [0, \ell] \to \mathbb{R}^n$ be a line segment and $c \in \mathbb{R}^n - \lambda$ and $r \in \mathbb{R}^+$. Then, $\kappa_{tot}^{abs}(I_{r,c}(\lambda)) = 2\angle(\lambda(0) - c, \lambda(\ell) - c)$.*

Proof. Let $\alpha = \angle(\lambda(0) - c, \lambda(\ell) - c)$, and assume $\alpha \neq 0$. Since $c \notin \lambda$, we know $\mathrm{I}_{r,c}(\lambda)$ is a subarc of a circle through c. Let β equal the angle of this subarc, and note that $\beta = \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\lambda))$. Euclidean geometry tells us that $\beta = 2\alpha$ when $\alpha \neq 0$, which gives $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\lambda)) = 2\angle(\lambda(0) - c, \lambda(\ell) - c)$.

Now say that $\alpha = 0$, and note λ is a segment of such a line through c. As the image under inversion of any line through an inversion's center is the same line we may conclude $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\lambda)) = 0 = 2\angle(\lambda(0) - c, \lambda(\ell) - c)$. $\square$

Proposition 3.24. *Let $p : [0, \ell] \to \mathbb{R}^2$ be a closed, convex, polygonal curve of n segments $\lambda_1, \ldots, \lambda_n$, where $p(t_i)$ is the initial point of λ_i. Let $r \in \mathbb{R}^+$. If $c \in \mathbb{R}^2$ is in the compact region bounded by the curve, then $\kappa_{tot}^{abs}(I_{r,c}(p)) = 6\pi$.*

Proof. Let $\alpha_i = \angle(p(t_i) - c, p(t_{i+1}) - c)$, and let β_i be the exterior angle at the joint $p(t_i)$. Notice that

$$\sum_{i=1}^{n} \beta_i = 2\pi \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 2\pi.$$

Let $\tilde{\beta}_i$ be the absolute value of the turning angle at $\mathrm{I}_{r,c}(p(t_i))$. (Inversions reverse orientation in $\mathbb{R}^2$, but we may disregard signs since we are only interested in total absolute curvature—see Figure 3.) As inversions are conformal, we know $\tilde{\beta}_i = \beta_i$, so

$$\sum_{i=1}^{n} \tilde{\beta}_i = 2\pi.$$

Lemma 3.23 tells us that each $\mathrm{I}_{r,c}(\lambda_i)$ contributes $2\alpha_i$ to $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(p))$, so the total contribution of all the $\mathrm{I}_{r,c}(\lambda_i)$ is

$$\sum_{i=1}^{n} \kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(\lambda_i)) = \sum_{i=1}^{n} 2\alpha_i = 4\pi.$$

Hence, $\kappa_{\mathrm{tot}}^{\mathrm{abs}}(\mathrm{I}_{r,c}(p)) = 2\pi + 4\pi = 6\pi$. $\square$

Acknowledgements

This research was supported in part by fellowships provided by the University of Iowa Graduate College. Declarations of interest: none.

References

[1] Borsuk, K. (1948). Sur la courbure totale des courbes fermées. *Annales de la Société Polonaise de Mathématique*, **20**:251-265.

[2] Cantarella, J., Kusner, R., & Sullivan, J. (2002). On the minimum ropelength of knots and links. *Inventiones Mathematicae*, **150**(2):257-286.

[3] do Carmo, M. (1992). *Riemannian geometry.* Boston: Birkhäuser.

[4] Fenchel, W. (1929). Uber krümmung and windung geschlossenen raumkurven. *Mathematische Annalen*, **101**:238-252.

[5] Freedman, M., He, Z., & Wang, Z. (1994). Möbius energy of knots and unknots. *Annals of Mathematics*, **139**:1-50.

[6] Litherland, R., Simon, J., Durumeric, O., & Rawdon, E. (1999). Thickness of knots. *Topology and Its Applications*, **91**:233-244.

[7] O'Hara, J. (1991). Energy of a knot. *Topology*, **30**(2):241-247.

[8] O'Hara, J. (2003). *Energy of knots and conformal geometry.* London: World Scientific.