MATH 4210
MIDTERM 2
April 13, 2018

NAME.______________________________SIGNATURE.________________________

Do all **5** problems, **20 points each**.

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. **Proofs must have logical continuity and must be mathematically correct**. You need to indicate or state any theorem that you use. The methods of proofs must be indicated, such as induction, proof by contradiction. Show all of your work or indicate its location in the space provided after each problem.

In this test, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural numbers, integers, rational numbers and real numbers, respectively.

You are allowed to use any theorem or proposition proved in class or in the textbook, unless the question is asking you to provide a proof of such a theorem. In this case, you may use any axiom, or proposition or theorem proven earlier, but you can neither use nor refer to the theorem you are proving or its consequences. Simply, it does **not** suffice to say ”This is a theorem we have in the book or we have done in class”. You are expected to provide a detailed proof. You can not refer to exercises, examples, or homework, unless you provide a solution to them. When you use a theorem, you can use its name (if name) such as Heine-Borel, or state the fact that you are using. Do not refer to a theorem number, since it is easy to make a mistake with those numbers.

No cell phones (and other communication devices) are allowed to be used during the exam.

DO NOT WRITE BELOW:

1._______

2. ______

3. ______

4. ______

5. ______ TOTAL.______

1. Provide the definitions of the following concepts in arbitrary metric spaces (X, d). The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which require a proof.

a. A **Cauchy sequence**

b. A **complete metric space**

c. An **absolutely convergent series** $\sum_{n=1}^{\infty} a_n$

d. A **continuous function** from one metric space into another

e. A **uniformly continuous function** from one metric space into another

2. State and prove the **Generalized Mean Value Theorem**.
HINT: It may contain an equation of the form

$$(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c)$$

a. Statement

b. Proof

3. If both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, prove that $\sum_{n=0}^{\infty} c_n$ converges absolutely, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N} \cup \{0\}$. You must provide a precise statement of every theorem that you use in your solution and we have proven in class or in the textbook.

4. **Prove that if** $f : (X, d_X) \to (Y, d_Y)$ **is a continuous function and** X **is compact**, **then** $f(X)$ **is compact**.

You are asked to prove Theorem 4.14 which we proved in class also. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 4.14 and the theorem we proved in class or their consequences. Simply, saying "This follows Theorem 4.14 and the theorems we proved in class" will not earn any credit. You are expected to provide proofs.

5. TRUE OR FALSE CIRCLE YOUR ANSWERS.
NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: **Read very carefully**.

TRUE FALSE **a**. Given a sequence $\{c_n\}$ in a compact metric space, every subsequence of $\{c_n\}$ converges to the same limit.

TRUE FALSE **b**. For given two real sequences $\{a_n\}$ and $\{b_n\}$, one has

$$\liminf_{n\to\infty} (a_n + b_n) \geq \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$$

TRUE FALSE **c**. If $f : (X, d_X) \to (Y, d_Y)$ is a continuous function, and $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.

TRUE FALSE **d**. If $f : (X, d_X) \to (Y, d_Y)$ is a continuous function, and $f(U)$ is open in Y, then U is open in X.

TRUE FALSE **e**. For every function $g : \mathbb{R} \to \mathbb{R}$ differentiable on all of $\mathbb{R}$, g' satisfies the intermediate value property even if g' is not continuous.