MATH 4210
MIDTERM 1
February 23, 2018

NAME.______________________________SIGNATURE.________________________

Do all **5** problems, **20 points each**.

Show all of your work in order to receive full credit. Every answer must be properly written with logically and grammatically correct sentences and mathematical expressions. **Proofs must have logical continuity and must be mathematically correct**. You need to indicate or state any theorem that you use. The methods of proofs must be indicated, such as induction, proof by contradiction. Show all of your work or indicate its location in the space provided after each problem.

In this test, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of natural numbers, integers, rational numbers and real numbers, respectively.

You are allowed to use any theorem or proposition proved in class or in the textbook, unless the question is asking you to provide a proof of such a theorem. In this case, you may use any axiom, or proposition or theorem proven earlier, but you can neither use nor refer to the theorem you are proving or its consequences. Simply, it does **not** suffice to say ”This is a theorem we have in the book or we have done in class”. You are expected to provide a detailed proof. You can not refer to exercises, examples, or homework, unless you provide a solution to them. When you use a theorem, you can use its name (if name) such as Heine-Borel, or state the fact that you are using. Do not refer to a theorem number, since it is easy to make a mistake with those numbers.

No cell phones (and other communication devices) are allowed to be used during the exam.

DO NOT WRITE BELOW:

1.______

2. ______

3. ______

4. ______

5. ______ TOTAL.______

1. Provide the definitions of the following concepts for a set E in an arbitrary metric space (X, d). The definitions you give need to be the same (or have the same meaning) as of those definitions given in the textbook or in class, without using a theorem or proposition which require a proof.

a. An **interior point** of a set E :

b. An **open set** E :

c. An **open cover** of a set K :

d. A **compact set** K :

2. Let E be a subset of a metric space (X, d).
a. **Prove that if $G \subset E$ and G is open then $G \subset E^\circ$, where E° is the interior of E.**

b. **Prove that $\partial E \cup E = \overline{E}$, where $\overline{E}$ is the closure of E, and ∂E is the boundary of E.**

3. Prove that $\mathbb{R}^k$ (with the metric $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$) has a countable and dense subset for all $k \in \mathbb{N}$ and $k \geq 2$. (You can use theorems from Chapter 1 for $k = 1$ provided that you state them.)

4. **Prove that every compact set in every metric space is closed and bounded**.

You are asked to prove Theorem 2.34 and another theorem we proved in class. You may use any axiom or proposition/theorem proven earlier, but you can neither use nor refer to Theorem 2.34 and the theorem we proved in class or their consequences. Simply, saying “This follows Theorem 2.34 and the theorems we proved in class” will not earn any credit. You are expected to provide proofs.

5. TRUE OR FALSE CIRCLE YOUR ANSWERS.
NO PARTIAL CREDITS. YOU ARE NOT EXPECTED TO SHOW WORK.
Correct answers are +4 points each,
wrong answers are -1 point each,
ambiguous answers are -2 points each, and
no answers are 0 point each.
Total of problem 5 will be added to your total grade only if it is positive.
HINT: **Read very carefully**.

TRUE FALSE **a**. Let S be an ordered set with the Greatest Lower Bound Property: that is, for every non-empty set $E \subset S$, if E is bounded below, then E has an infimum in S. Then, it is not necessary that a bounded set E has a supremum in S, unless $S = \mathbb{R}$ with the usual order.

TRUE FALSE **b**. There exists no set E which is both open and compact in an arbitrary metric space (X, d).

TRUE FALSE **c**. The set of all functions $f : \{0,1\} \to \mathbb{N}$ is a countable set, where $\mathbb{N}$ denotes the Natural numbers.

TRUE FALSE **d**. For $x,y \in \mathbb{R} - \{0\}$, the set of all nonzero real numbers, $d_0(x,y)$ is a metric where $d_0(x,y) = \begin{cases} |x-y| & \text{if } xy > 0 \\ |x-y| + 1 & \text{if } xy < 0 \end{cases}$.

TRUE FALSE **e**. The set $E = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ is a connected set in $\mathbb{R}^2$ with respect to the standard metric.