Defn Let (X, d) be a metric space, $E \subseteq X$.

$\bar{E} = E \cup E'$ is called the closure of E.

2.27. **Thm:** (a) $\bar{E}$ is a closed set.

(b) $E = \bar{E} \iff E$ is closed.

(c) $\bar{E} \subset F$, for every closed set F s.t. $E \subseteq F$.

Proof: (a) To show $\bar{E}$ is closed

(WTS) $\left\{ \begin{array}{l} \forall p \in (\bar{E})^c \ \exists r > 0 \quad N_r(p) \subseteq (\bar{E})^c \\ \text{i.e. } (\bar{E})^c \text{ is open} \end{array} \right.$

Let $p \in (\bar{E})^c = (E \cup E')^c = E^c \cap (E')^c$

$p \in E^c$

$p \in (E')^c$

$p \notin E$ and p is not a limit pt of E $(p \notin E')$

$p \notin E$ and not $(\forall r > 0 \quad N_r(p) \cap (E - \{p\}) \neq \emptyset)$

$p \notin E$ and $\exists r > 0 \quad N_r(p) \cap (E - \{p\}) = \emptyset$.

$\exists r > 0 \quad N_r(p) \cap E = \emptyset$. $(p \notin E)$

$N_r(p) \subseteq E^c$

$\left(\begin{array}{l} E \subseteq \bar{E} \\ E^c \supseteq (\bar{E})^c \end{array} \right)$

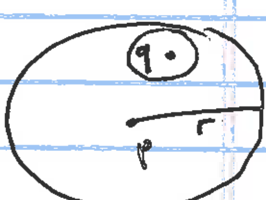
$\bar{E} = E \cup E'$

Caution: Need to show more:

Suppose $N_r(p) \cap E' \neq \emptyset$.

$\exists q \in N_r(p) \cap E'$

$\exists \delta > 0 \quad N_\delta(q) \subseteq N_r(p)$



$q \in E'$

$N_\delta(q) \cap E - \{q\} \neq \emptyset$

$N_\delta(q) \cap E \neq \emptyset$ Contradiction.

So $N_r(p) \cap E' = \emptyset$

$N_r(p) \subseteq E^c \implies N_r(p) \cap E = \phi.$

$N_r(p) \cap E' = \phi.$

$$N_r(p) \cap (\underbrace{E \cup E'}_{\bar{E}}) = \phi$$

$N_r(p) \subseteq (\bar{E})^c$ Done.

(b) (WTS) $E = \bar{E} \iff E$ is closed

<u>proof:</u> $E = \bar{E} \iff E = E \cup E' \iff E' \subseteq E \iff E$ is closed

(c) (WTS) F closed $E \subseteq F \implies \bar{E} \subseteq F$

<u>proof:</u> $E \subseteq F \implies E' \subseteq F'$

$p \in E'$: $\forall r > 0\ N_r(p) \cap (E - \{p\}) \neq \phi \Rightarrow \forall r > 0\ N_r(p) \cap (F - \{p\}) \neq \phi$ $p \in F'$

F closed $F' \subseteq F$.

$\bar{E} = E \cup E'$

$$\underset{\uparrow \text{defn}}{\bar{E} =} E \cup E' \underset{\uparrow \text{above}}{\subseteq} F \cup F' \underset{\text{defn}}{=} \bar{F} \underset{(b)}{=} F \leftarrow \text{closed}$$

Prop 2.27.5 Let $E \subseteq X$, a metric space.

we'll do
(a) (i) $E \cup \partial E = \bar{E}$ $(= E \cup E')$
(ii) $\partial E \subseteq E \iff E$ is closed

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(b)(i) $\partial E \cap \operatorname{int} E = \emptyset$
(ii) E open $\iff E \cap \partial E = \emptyset$.
(iii) $\bar{E} = (\operatorname{int} E) \cup (\partial E)$

Proof of (a)

(i) wts $E \cup \partial E = E \cup E'$ $(= \bar{E})$

wts $E \cup \partial E \subseteq E \cup E'$

Let $p \in E \cup \partial E$. If $p \in E$, we're done

If $p \notin E$, then I need $p \in E'$.

$p \notin E \Rightarrow p \in \partial E$

$\Rightarrow \forall r > 0$ $N_r(p) \cap E \neq \emptyset$

$N_r(p) \cap E^c \neq \emptyset$ $p \notin E$

$\forall r > 0$ $N_r(p) \cap (E - \{p\}) \neq \emptyset$

$p \in E'$.

We showed $E \cup \partial E \subseteq E \cup E'$.

wts: $E \cup \partial E \supseteq E \cup E'$

Let $p \in E \cup E'$ be an arbitrary pt.

if $p \in E$, then we're done

If $p \notin E$, $p \in E'$, then I need $p \in \partial E$.

$$p \in E' \implies \forall r > 0 \quad N_r(p) \cap (E - \{p\}) \neq \phi.$$
$$\forall r > 0 \quad N_r(p) \cap E \neq \phi \quad (p \notin E)$$

$$\forall r > 0 \quad N_r(p) \cap \underbrace{E^c}_{\text{has } p.\ (p \notin E)} \neq \phi$$

$$p \in \partial E$$

$$\forall p \in E \cup E', \quad p \in E \cup \partial E.$$
$$E \cup E' \subseteq E \cup \partial E$$
$$\bar{E} = E \cup E' = E \cup \partial E$$

(a) (ii)

(WTS) $\partial E \subseteq E \iff E$ is closed

Proof. $\partial E \subseteq E \iff E \cup \partial E = E \iff E \cup E' = E \cup \partial E = E$
$\iff E \cup E' = E$
$\iff E' \subseteq E$
$\iff E$ is closed

as in (b) of Prop 2.27.

Caution: $E \cup \partial E = E \cup E' \not\Rightarrow \partial E = E'$ or $\partial E \subseteq E'$ or $E' \subseteq \partial E$

none hold.

Ex $E = (0,1) \cup \{2\}$, $(\mathbb{R}, \|\cdot\|)$
$\partial E = \{0, 1, 2\}$
$E' = [0, 1]$

Thm 2.28 Let $E \subseteq \mathbb{R}$, E bounded above

ϕ^{**} so that $y = \sup E \in \mathbb{R}$, then

$$y \in \partial E \subseteq \overline{E}$$

(ii) If E is closed, then $y = \sup E \in E$

Corollary: If E is $\subseteq \mathbb{R}$, closed and bounded then

$\inf E \in E$

and $\sup E \in E$.

Ex: Is Converse True? $\inf E \in E$, $\sup E \in E \overset{?}{\Rightarrow}$ | E is closed and bounded.

Ex $E = [0,1) \cup (2,3] \subseteq \mathbb{R}$, $\|\cdot\|$ standard

$0 = \inf E \in E$

$3 = \sup E \in E$

E bounded, but

E not closed.