Prop 2.20 Let $E \subseteq (X, d)$

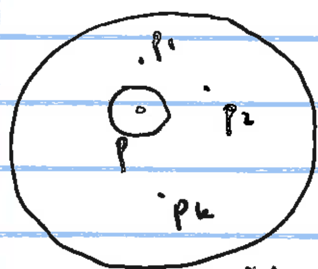
If $p \in E'$, then every neighborhood of p contains infinitely many elements of E.

Proof Suppose not ($\forall r > 0$ $N_r(p) \cap E$ is infinite)

$\exists r_0 > 0$ $N_{r_0}(p) \cap E$ is finite.

Let $\{p_1, p_2, \ldots p_n\}$ be set of all $N_{r_0}(p) \cap E$

$r = \min(d(p,p_1), d(p,p_2), \ldots d(p,p_n), r_0) > 0$



$N_r(p) \cap \{p_1, p_2, \ldots p_n\} = \emptyset$

since $d(p, p_i) \geq r$

$N_r(p) \cap E = N_r(p) \cap N_{r_0}(p) \cap E = N_r(p) \cap \{p_1, \ldots p_n\} = \emptyset$

$r \leq r_0$.

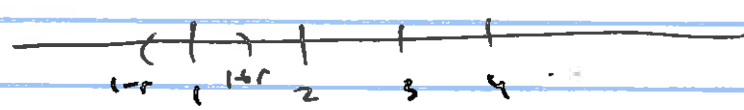
contradicts defn of a limit pt.

For any metric space }

Corollary: For any finite set E, $E' = \emptyset$.

For any finite set E, E is closed. ($E' \subseteq E$, $E' = \emptyset$)

Ex ① $\mathbb{N} \subseteq \mathbb{R}$, $d = \| \cdot \|$



$\mathbb{N}$ is not open in $\mathbb{R}$, since

$\exists$ no $r > 0$ s.t. $(1-r, 1+r) \subseteq \mathbb{N}$.

$(1-r, 1+r) = N_r(1)$.

No limit pts $\Updownarrow$ Def. (Discrete) $\longrightarrow$ ② $\mathbb{N}' = \emptyset$: since for $r = \frac{1}{2}$ $\forall n \in \mathbb{N}$ $N_{\frac{1}{2}}(n) \cap \mathbb{N} = \{n\}$ (+ prop 2.20)

Ex ③

discrete metric on $\mathbb{R}$ $\quad d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$N_{\frac{1}{2}}(p) = \{p\}$ open wrt discrete metric

Every set $E \subseteq (\mathbb{R}, \text{discrete metric})$ is open

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Lemma: Let $\{E_\alpha \subseteq X \mid \alpha \in A\}$ be a collection of subsets of X. Then

$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$$

Proof: $x \in \left(\bigcup_\alpha E_\alpha\right)^c \iff x \in X$ and $x \notin \left(\bigcup_\alpha E_\alpha\right)$

$\iff x \in X$ and not$(\exists \alpha \in A,\ x \in E_\alpha)$

$\iff x \in X$ and $\forall \alpha \in A$ $x \notin E_\alpha$.

$\iff \forall \alpha \in A$ $(x \in X$ and $x \notin E_\alpha)$

$\iff \forall \alpha \in A$ $(x \in E_\alpha^c)$

$\iff x \in \bigcap_{\alpha \in A} (E_\alpha)^c$.

Prop Let $E \subseteq (X, d)$, metric space

E^c is closed $\iff$ E is open

Proof:

(WTS) (i) E open $\Longrightarrow$ E^c is closed.

Assume E is open i.e. $\text{int}\, E = E$ (Hypothesis)

(WTS) $(E^c)' \subseteq E^c$

Let q be any element of $(E^c)'$

$\forall r > 0 \quad N_r(q) \cap (E^c - \{q\}) \neq \phi$.

$\forall r > 0 \quad N_r(q) \cap E^c \neq \phi$.

$\forall r > 0 \quad N_r(q) \not\subseteq E$

$q \notin E$ (since $\text{int}\, E = E$ implies $\forall p \in E \Rightarrow \exists r > 0 \; N_r(p) \subseteq E$)

$q \in E^c$.

So, $\forall q \in (E^c)'$, we have $q \in E^c$

$(E^c)' \subseteq E^c$

E^c closed.

WTS (ii) E^c is closed $\Longrightarrow$ E is open.

Let $p \in E$.

$p \notin E^c$

$p \notin (E^c)'$ $\quad$ (E^c closed: $(E^c)' \subseteq E^c$)

not $(\forall r > 0 \; N_r(p) \cap (E^c - \{p\}) \neq \phi)$

$\exists r > 0 \quad N_r(p) \cap E^c - \{p\} = \phi$.

$\exists r > 0 \quad N_r(p) \cap E^c = \phi$. $\quad (p \notin E^c)$

$\exists r > 0 \quad N_r(p) \subseteq E$ $\quad$ (so E is open.)

(2.24) **Thm:** Let $G_\alpha, F_\alpha \subseteq X$, a metric space

(a) For any collection $\{G_\alpha\}_{\alpha \in A}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.

(b) For any finite collection $\{G_i\}_{i=1}^n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.

(c) For any collection $\{F_\alpha\}_{\alpha \in A}$ of closed sets,

$$\bigcap_{\alpha \in A} F_\alpha \text{ is closed.}$$

(d) For any finite collection $\{F_i\}_{i=1}^n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed

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Proof: (a) Let $p \in \bigcup_{\alpha \in A} G_\alpha$.

$$p \in G_{\alpha_0} \text{ for some } \alpha_0 \in A$$

$$\exists r > 0 \quad N_r(p) \subseteq G_{\alpha_0} \subseteq \bigcup_{\alpha \in A} G_\alpha.$$

(b) Let $p \in \bigcap_{i=1}^n G_i$

$$\forall i = 1, 2, \ldots n \quad p \in G_i$$

$$\forall i = 1, \ldots n \ \exists r_i > 0 \qquad N_{r_i}(p) \subseteq G_i$$

$$r = \min(r_1, r_2, \ldots r_n) > 0, \quad B_r(p) \subseteq G_i \ \forall i$$

$$B_r(p) \subseteq \bigcap_{i=1}^n G_i.$$

(c) Let $\{F_\alpha\}_{\alpha \in \Lambda}$ be any collection of closed sets.

F_α closed $\Rightarrow$ F_α^c is open.

$$\Big(\bigcap_{\alpha \in A} F_\alpha\Big)^c = \underbrace{\bigcup_{\alpha \in A} F_\alpha^c}_{\text{open by (a)}} \qquad \text{(By Lemma)}$$

$\Rightarrow \bigcap_{\alpha \in A} F_\alpha$ is closed

(d) HW by Lemma + (b)

Ex

① $$\bigcap_{n=1}^{\infty} \underbrace{\left(-1-\tfrac{1}{n},\ 1+\tfrac{1}{n}\right)}_{\text{each open}} = \underbrace{[-1,1]}_{\text{not open}}. \qquad (\mathbb{R}, \|\cdot\|)$$

$$\bigcup_{n=2}^{\infty} \underbrace{\left[\tfrac{1}{n},\ \tfrac{n}{n+1}\right]}_{\text{each closed}} = \underbrace{(0,1)}_{\text{not closed}}.$$

② (i) $\{x\}$ is closed in any metric space $X \ni x$.

Any set $E = \bigcup_{x \in E} \{x\}$ is a union of closed sets.

(ii) $\{x\}$ closed $\Rightarrow$ $\{x\}^c$ is open.

Let F be an arbitrary set.

Let $F = E^c = \left(\bigcup_{p \in E} \{p\} \right)^c = \bigcap_{p \in E} \{p\}^c$

$$= \bigcap_{\substack{p \in \mathcal{I} \\ p \notin F}} \{p\}^c = F.$$

Every set F is an intersection of open sets.