<u>Defn</u> Let (X, d) be a metric space.
We define the following, for $E \subseteq X$

- <u>Neighborhood</u>:

$$N_r(p) = \{ q \in X \mid d(q,p) < r \}$$

r = radius, p = center

- A point p is called a <u>limit point</u> of $E \subseteq X$ if

$$\forall r > 0 . \quad N_r(p) \cap (E - \{p\}) \neq \emptyset .$$

In other words: every neighborhood of p has pts from E other than p.

Notation E' = the set of all limit pts of E.

- A point $p \in E$ is called an <u>isolated pt</u> if $p \notin E'$

- A set E is called <u>closed</u> if $E' \subseteq E$.

- For a set E, a point p is called an <u>interior pt of E</u> if $\exists r > 0$ s.t. $N_r(p) \subseteq E$.

Notations: $E^\circ = \mathrm{int}(E)$ is the set of all interior points of E. ($E^\circ \subseteq E$ always)

- A set E is called <u>open</u> if $E = \mathrm{int}\, E = E^\circ$

- Complement of E in X, $E^c = X - E$

$$= \{ p \in X \mid p \notin E \}$$

- For a set E, a point $p \in X$ is called a boundary pt of E if
  $\forall r > 0 \ (N_r(p) \cap E \neq \emptyset$ and $N_r(p) \cap E^c \neq \emptyset)$

$$\Longleftrightarrow (\forall r > 0 \ N_r(p) \cap E \neq \emptyset \text{ and } \forall r > 0 \ N_r(p) \cap E^c \neq \emptyset)$$

  Notation: ∂E denotes the set of all boundary pts of E.

- $E \subseteq X$ is called perfect if
  - E is closed
  and
  - $E \subseteq E'$

  $(\Longleftrightarrow E = E'.)$

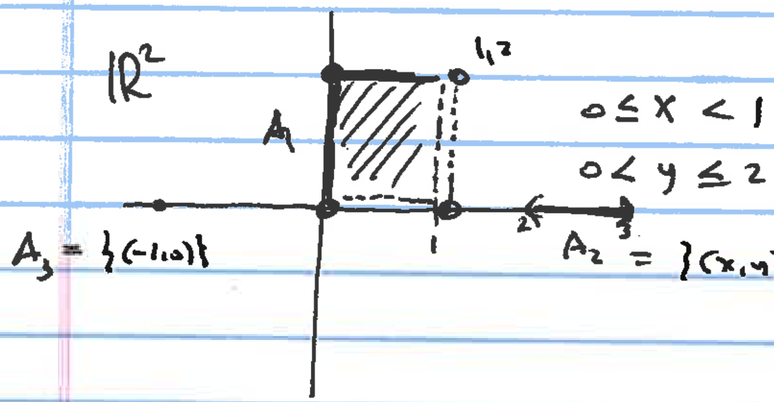
- A set $E \subseteq X$ is called bounded if $\exists M \in \mathbb{R}$ s.t.

$$E \subseteq N_M(p) \text{ for some } p \in X.$$

- A set E is called dense in X if

$$X = E \cup E'$$

Examples (no proofs yet) (Later we will do examples with proofs)



$A = A_1 \cup A_2 \cup A_3 \subseteq \mathbb{R}^2$, $\|x-y\|$ standard metric

$\operatorname{Int} A = \{(x,y) \mid 0 < x < 1,\ 0 < y < 2\}$.

Caution $p_0 = (2.5, 0) \notin \operatorname{int} A$ since

$\forall r > 0$ $N_r(p_0)$ contains points with non-zero y-coordinates.

$N_r(p_0) \not\subseteq A_2$ for any r.

A open? No $p_0 = (2.5, 0) \in A$ but $p_0 \notin A^\circ$.

A closed?

What is A'? $A' = \left\{(x,y) \,\middle|\, \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 2 \end{array}\right\} \cup \left\{(x,y) \,\middle|\, \begin{array}{l} 2 \le x \le 3 \\ y = 0 \end{array}\right\}$

$A' \not\subseteq A \Rightarrow A$ not closed

$\{(-1,0)\}$ is an isolated pt of A

A is bounded: $N_4((0,0)) \supset A$

$$(\text{Boundary of } A) = \partial A$$

$$\partial A = \left\{ (x,y) \,\middle|\, \begin{pmatrix} 0 \le x \le 1 \\ y = 0 \end{pmatrix} \text{ OR } \begin{pmatrix} 0 \le x \le 1 \\ y = 2 \end{pmatrix} \text{ OR } \begin{pmatrix} x = 0 \\ 0 \le y \le 2 \end{pmatrix} \text{ OR } \begin{pmatrix} x = 1 \\ 0 \le y \le 2 \end{pmatrix} \text{ OR } \begin{pmatrix} 2 \le x \le 3 \\ y = 0 \end{pmatrix} \text{ OR } \begin{pmatrix} x = -1 \\ y = 0 \end{pmatrix} \right\}$$

__Examples__ · Finite sets E, $E' = \emptyset$.

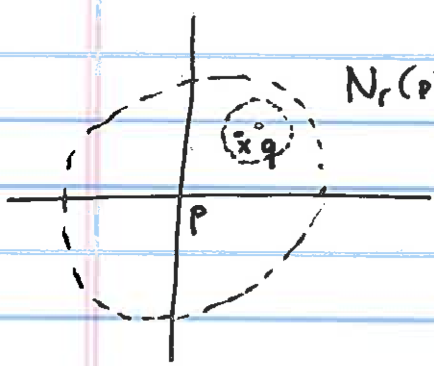
E is closed; $E' = \emptyset \subseteq E$.

· $\mathbb{R}^2, \emptyset \subseteq (\mathbb{R}^2, \|\cdot\|)$

both open and closed, $\partial = \emptyset$.

$\mathbb{R}^2$ perfect, $(\mathbb{R}^2)' \subseteq \mathbb{R}^2$ [Actually $(\mathbb{R}^2)' = \mathbb{R}^2$].

__Thm 2.19__ Every neighborhood is an open set. (In every metric space)



__Proof__: Let $N_r(p)$ be given, $r > 0$, $p \in X$.

Let q be any pt of $N_r(p)$, we need to show $\exists\, \delta > 0$ s.t.

$$N_\delta(q) \subseteq N_r(p)$$

$$q \in N_r(p) \Rightarrow d(p,q) < r$$

Let $\delta = r - d(p,q) > 0$

__WTS__: $N_\delta(q) \subseteq N_r(p)$

Let $x \in N_\delta(q)$ be an arbitrary pt.

We Have: $\left.\begin{array}{l} d(x,q) < \delta \\ d(p,q) < r \end{array}\right\}$ and WTS: $d(x,p) < r$.

$$d(x,p) \le \underbrace{d(p,q)}_{d(p,q) = r-\delta} + \underbrace{d(q,x)}_{<\delta} < r - \delta + \delta = r$$

Hence $B_\delta(q) \subseteq B_r(p)$.

since we showed that

$\forall x \in B_\delta(q)$, one has $x \in B_r(p)$.