Jan 29, 2018

①

(CANTOR) **Thm:** Let A be the set of all sequences whose elements are 0 or 1. Then A is uncountable.

Remark: ① $f: A_0 \to B_0$

A_0: n elements; B_0: m elements

How many such f are there? m^n

$A_0 = \{a_1, a_2, \dots a_n\}$

$B_0 = \{b_1, b_2 \dots b_m\}$

$f(a_i) = $ (m choices $b_1 \dots b_m$)

Do for each $i = 1, \dots n$.

$$\underbrace{m \cdot m \cdot \dots m}_{n} = m^n.$$

Some books $B_0^{A_0} = \{f \mid f: A_0 \to B_0\}$.

②

$$A = \{s \mid s: \mathbb{N} \to \{0,1\}\} = \{0,1\}^{\mathbb{N}}$$

Proof Thm: Let E be a countable subset of $A = \{s \mid s: \mathbb{N} \to \{0,1\}\}$

$\exists\, f: \mathbb{N} \to E$ bijection

$f(1) = s_1$ sequence in 0,1

$\vdots$

$f(i) = s_i$ " in 0,1

$\forall k \in \mathbb{N}$

$$\left.\begin{array}{ll} s_1 : & s_{11}, s_{12}, s_{13}, s_{14}, \ldots \\ s_2 : & s_{21}, s_{22}, s_{23}, s_{24} \\ s_3 : & s_{31}, s_{32}, s_{33}, s_{34} \ldots \\ \vdots & \\ s_k : & s_{k1}, s_{k2}, s_{k3}, \ldots \\ \vdots & \end{array}\right\} \quad s_{ij} = 0 \text{ or } 1.$$

One creates a new sequence $t : t_1, t_2 \ldots t_k \ldots$

$$t_k = \begin{cases} 0 & \text{if } s_{kk} = 1 \\ 1 & \text{if } s_{kk} = 0 \end{cases}$$

$t \neq$ all of s_i

$t \notin E$.

$\forall$ countable subset E of A, $E \neq A$.

$$\left.\begin{array}{l} A \text{ can't be countable.} \\ A \text{ can't be finite} \end{array}\right\} \Rightarrow A \text{ is uncountable.}$$

Why?

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\delta_1 : 1, 0, 0, \ldots\ 0,$

$\delta_2 : 0, 1, 0, 0, \ldots$

$\delta_3 : 0, 0, 1, \ldots$ the k^{th}

$\delta_k : 0, 0, \ldots\ 0, 1, 0, \ldots$

$\delta_i \in A \ \forall i \in \mathbb{N}$

$\delta_i \neq \delta_j$ if $i \neq j$

$\Rightarrow A$ infinite

Corollary: $\mathbb{R}$ is uncountable

$\forall r \in [0,1]$ $\exists$ binary expansion

$r = 0.1001101100 0 1\ldots \overset{1-1}{\longleftrightarrow} s = \underbrace{(1,0,0,1,1,0,1,1,0\ldots}_{\text{sequence}}$

$r = \frac{1}{2} + \frac{0}{4} + \frac{0}{8} + \frac{1}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128} + \ldots$

Caution $\quad 0.\overline{1} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1 = 1.0$

Some real numbers have 2 different binary expansions, but not 3 for any $r \in [0,1]$.

If $[0,1]$ were countable then $A = \{ s \mid s : \mathbb{N} \to \{0,1\} \}$ would be countable

$\downarrow$ but not countable

$\Rightarrow [0,1]$ uncountable

$\Rightarrow \mathbb{R}$ uncountable

(by Thm: "subsets of countable sets are countable or finite")

METRIC SPACES:

Defn A set X with a function $d : X \times X \to [0,\infty)$ is called a metric space if

(i) $d(p,q) > 0 \quad \forall\, p,q \in X,\ p \neq q$
and $d(p,p) = 0 \quad \forall\, p \in X$

(ii) $d(p,q) = d(q,p) \quad \forall\, p,q \in X$

(iii) $d(p,q) \leq d(p,r) + d(r,q) \quad \forall\, p,q,r \in X$.

Ex ① $\mathbb{R}^n$

a) $d(x,y) = \|x-y\|$ standard Euclidean distance

b) $d_1((x_1, \dots x_n),(y_1, \dots y_n)) = \sum_{i=1}^{n} |x_i - y_i|$

c) $d_\infty((x_1, \dots x_n),(y_1, \dots y_n)) = \max_{1 \leq i \leq n} |x_i - y_i|$

d) Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be any injective map.

$$d_A(x,y) = \|A(x) - A(y)\|$$

d_A is the pull back metric of $\|\ \|$ in $\mathbb{R}^m$ via A into $\mathbb{R}^n$.

Ex

Let $\mathscr{C}[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$

$$d_\infty(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$$

$$d_1(f,g) = \int_a^b |f(x) - g(x)|\,dx$$

$$d_2(f,g) = \left(\int_a^b |f(x) - g(x)|^2 \right)^{1/2}$$

Ex On any set X

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$ discrete metric

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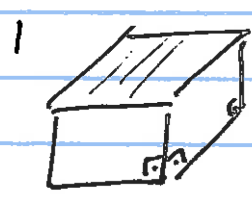
Defn $(a,b) = \{ x \in \mathbb{R} \mid a < x < b \}$ segments

$[a,b] = \{ x \in \mathbb{R} \mid a \le x \le b \}$

Cell Let $a_i < b_i \quad \forall i = 1, \dots n$

$\{ \underbrace{(x_1, x_2, \dots x_n)}_{\in \mathbb{R}^n} \mid a_1 \le x_1 \le b_1 ;\ a_2 \le x_2 \le b_2, \dots, a_n \le x_n \le b_n \}$



$n = 3$

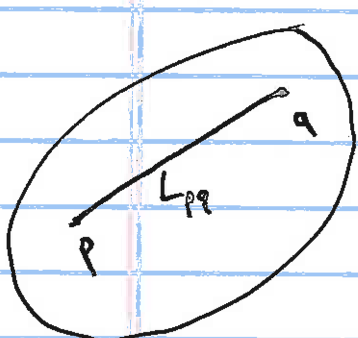
- Balls in $\mathbb{R}^k$, standard metric

$\forall p \in \mathbb{R}^k$ $\quad B_r(p) = \{ q \in \mathbb{R}^k \mid |q-p| < r \}$ open ball.

$\forall r \in \mathbb{R}$ $\quad \overline{B}_r(p) = \{ q \in \mathbb{R}^k \mid |q-p| \le r \}$ closed ball.

- A set $E \subseteq \mathbb{R}^k$ is called convex if

$\forall p,q \in E$, the line segment L_{pq} joining p to q is also in E



$$\{ \lambda p + (1-\lambda) q \mid 0 \le \lambda \le 1 \}$$

Prop: $B_r(p)$, $\overline{B}_r(p)$ are convex. HW to read.

HW