Jan 26, 2018

①

Countable + Uncountable Sets :

Let A, Ω be sets s.t.

$\forall \alpha \in A, \; E_\alpha \subseteq \Omega$

Notation $x \in \bigcup_{\alpha \in A} E_\alpha \iff \exists \alpha \in A$ s.t. $x \in E_\alpha$

$x \in \bigcap_{\alpha \in \Lambda} E_\alpha \iff \forall \alpha \in A \; x \in E_\alpha$

If $A = J_n = \{1, 2, 3, \ldots n\}$ $\qquad \bigcup_{m=1}^{n} E_m$

If $A = \mathbb{N}$ $\qquad \bigcap_{m=1}^{\infty} E_m$

Ex $\quad \bigcap_{m=1}^{\infty} \left(-\frac{1}{m}, \frac{1}{m}\right) = \{0\}$

Arch. Prop. $\forall x > 0 \; \exists m \in \mathbb{N} \quad 0 < \frac{1}{m} < x$

i.e $1/x < m \in \mathbb{N}$.

$$\bigcup_{m=1}^{\infty} \left[0, \frac{m}{m+1}\right] = [0, 1)$$

2.12 Thm Let $\{E_n\}$ be a sequence of countable sets. Then

$$S = \bigcup_{n=1}^{\infty} E_n \quad \text{is a countable set.}$$

Proof: For each i $E_i \sim \mathbb{N}$.

" " " let $f_i : \mathbb{N} \to E_i$ be a bijection

$$f_i(n) = x_{in}$$

$$\begin{array}{lll}
E_1 & x_{11}, x_{12}, x_{13} \ldots & \text{Sequence exhausting } E_1 \\
E_2 & x_{21}, x_{22}, x_{23} \ldots & \quad " \quad " \quad E_2 \\
\vdots & x_{31} & \\
E_k & x_{k1}, x_{k2}, x_{k3}, \ldots & \quad " \quad " \quad E_k
\end{array}$$

$$x_{11};\ x_{21}, x_{12};\ x_{31}, x_{22}, x_{13};\ x_{41}, x_{32}, x_{23}, x_{14};\ x_{51} \ldots$$

renumber sequence

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

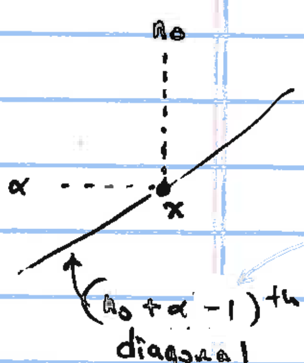
$$y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 \quad y_6 \quad y_7 \quad \ldots$$

Claim: $\{y_k\}$ exhausts S:

$$\forall x \in S \quad \exists\, \alpha \in A = \mathbb{N},\ x \in E_\alpha$$

$$\exists \quad f_\alpha : \mathbb{N} \to E_\alpha \text{ onto}$$

$$n_0 \dashrightarrow x$$



$$\exists\, n_0 \in \mathbb{N} \quad f_\alpha(n_0) = x$$

$$\exists\, k \text{ s.t. } \quad y_k = x \qquad \left(k \leq \frac{(\alpha+n_0)(\alpha+n_0-1)}{2} \right)$$

Hence: $S = \{ y_n \mid n \in \mathbb{N} \}$

Let $g : \mathbb{N} \to S$ onto, but not necessarily 1-1
$n \longmapsto y_n$

Typo

Let $T = \{ k \in \mathbb{N} \mid g(k) \in S, \forall n_{\in \mathbb{N}} \, (k > n, g(k) \neq g(n) \}$

$T \subseteq \mathbb{N}$.

T is countable or finite (Thm 2.8)

$g|T : T \to S$ onto, & 1-1

$\forall x \in S, \quad g^{-1}(x) = \{ k_1, \ldots k_n, \ldots \} \subseteq \mathbb{N}$
only the min of $k_1, \ldots, k_n$ will be in T

$\Rightarrow S$ is countable or finite.

$E_1 \subseteq S = \bigcup E_i$, E_1 is infinite, so: S is infinite

$\Rightarrow S$ is countable.

2.13 Thm: Let A be countable.

Then $A^n = A \times \cdots \times A = \{ (x_1, x_2, \ldots x_n) \mid x_i \in A, i = 1, \ldots n \}$ is countable.

Better to use a_i instead of x_i

Proof: Let $a_1, a_2, \ldots a_n \ldots$ be a sequence exhausting A:
$f : \mathbb{N} \to A$ bijection
$f(i) = a_i$

$$A^2 = \{ (x, y) \mid x \in A, y \in A \} = \bigcup_{i=1}^{\infty} \{ (a_i, a_j) \mid j = 1, 2, 3, \ldots \}$$

For each fixed i: $\{(a_i, a_j) \mid j = 1, \dots n, \dots\}$ is countable $j \in \mathbb{N}$

$i = 1, 2, 3 \dots\dots$ countable indexing set.

$$\left.\bigcup_{i=1}^{\infty} \underbrace{\{(a_i, a_j) \mid j = 1, \dots n, \dots\}}_{\text{countable}}\right\} \text{countable by the previous Thm. 2.12}$$

countably many i

$$\underset{\text{countable}}{A^{k+1}} = \underset{\text{countable}}{A} \times \underset{\text{and countable}}{A^k}$$

Rest follows by using induction

Corollary: $\mathbb{Q}$ is countable

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Z} - \{0\}} \underbrace{\left\{ \tfrac{p}{q} \mid p \in \mathbb{Z} \right\}}_{\text{each countable for a fixed } q.}$$

countable indexing set

By Thm. 2.12 $\mathbb{Q}$ is countable