May 2, 2018

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compare

Continuity: Given f, Given x $\forall \varepsilon > 0$ $\exists \delta > 0$ depends on f, x, ε

$\forall y \in E$ $(|f(x) - f(y)| < \varepsilon \Leftarrow |x-y| < \delta)$

Uniform C.: Given f $\forall \varepsilon > 0$ $\exists \delta > 0$ depends f, ε

$\forall x, y \in E$ $(|f(x) - f(y)| < \varepsilon \Leftarrow |x-y| < \delta)$

Equicontinuity: Given $\mathcal{F}$ $\forall \varepsilon > 0$ $\exists \delta > 0$ depends on $\varepsilon, \mathcal{F}$

$\forall x, y \in E$ $\forall f \in \mathcal{F}$ $|f(x) - f(y)| < \varepsilon \Leftarrow |x-y| < \delta$

7.24 **Thm:** Let K be a compact metric space.

$\forall n$ $f_n : K \to \mathbb{R}$, $f : K \to \mathbb{R}$ be all continuous.

If $f_n \to f$ uniformly on K then

$\{f_n : n \in \mathbb{N}\}$ is an equicontinuous family.

Proof: Since $f_n \to f$ uniformly, it is uniformly Cauchy.

Let $\varepsilon > 0$ $\exists N$ $\forall n, m \geq N$ $\|f_n - f_m\| \leq \frac{\varepsilon}{3}$

Particularly $\forall n \geq N$ $\|f_N - f_n\| \leq \frac{\varepsilon}{3}$.

K compact $\Rightarrow$ $f_i : K \to \mathbb{R}$ uniformly continuous, $i = 1, 2, \ldots N$

$\forall i = 1, 2, 3, \ldots N$ $\exists \delta_i > 0$ $\forall x, y \in K$

$d(x,y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| \leq \frac{\varepsilon}{3} < \varepsilon$

Let $\delta = \min(\delta_1, \delta_2, \ldots \delta_N) > 0$

$\forall x, y \in K$ $\forall i = 1, \ldots N$ $d(x,y) < \delta \Rightarrow |f_i(x) - f_i(y)| \leq \frac{\varepsilon}{3}$

$\forall n > N$

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)|$$

$$\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|$$

$$\leq \|f_n - f_N\| + \varepsilon/3 + \|f_N - f_n\| \leq \varepsilon.$$

#

* **7.25 Thm:** Let K be compact

$\forall n \; f_n \in \mathscr{C}(K)$

$\{f_n\}_{n=1}^{\infty}$ is pointwise bdd & equicontinuous

Then

(i) $\{f_n\}_{n=1}^{\infty}$ is uniformly bdd

(ii) $\{f_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence

Proof (i) Let $\varepsilon > 0$ $\exists \delta > 0$

$\forall x, y \in K, \; \forall n \in \mathbb{N} \quad |f_n(x) - f_n(y)| < \varepsilon$

$|x-y| < \delta$

$$K \subseteq \bigcup_{p \in K} N_\delta(p)$$

K compact $\Rightarrow$ $\exists$ finite subcover $K \subseteq \bigcup_{i=1}^{\ell} N_\delta(p_i)$

$\{f_n\}$ is pointwise bdd on each p_i

$$\sup_{n \in \mathbb{N}} |f_n(p_i)| = M_i < \infty \qquad i = 1, 2, \ldots \ell$$

Let $M = \max(M_1, M_2, \ldots M_\ell) + \varepsilon < \infty$

$\forall x \in K \subseteq \bigcup_{i=1}^{k} N_\delta(p_i) \qquad x \in N_\delta(p_{i_0})$ for some i_0

$d(x, p_{i_0}) < \delta$

$\Rightarrow \forall n \quad |f_n(x) - f_n(p_{i_0})| < \varepsilon$

$|f_n(x)| \le |f_n(p_{i_0})| + \varepsilon \le M_{i_0} + \varepsilon \le M$. #

(ii) K compact, choose E countable $\subseteq K$

$\overline{E} = K$

(Exc 2.25)

Look at $\{f_n\}$ on E.

(Thm 7.23)

There is a subsequence f_{n_k} of f_n s.t.

f_{n_k} is convergent on E. Let $g_i = f_{n_i}$, $i \in \mathbb{N}$

WANT g_i converges uniformly on K.

$$K \subseteq \bigcup_{x \in E} N_\delta(x) \quad \left(\text{otherwise any } x \in K - \bigcup_{x \in E} N_\delta(x) \right.$$

Caution

$$\Rightarrow N_{\frac{\delta}{2}}(x) \cap E - \{x\} = \emptyset.$$

$$\left. \Rightarrow x \notin \overline{E} \quad \text{X contradiction.} \right)$$

$$K \text{ compact} \Rightarrow K \subseteq \bigcup_{i=1}^{l_0} N_\delta(x_i); \quad x_i \in E, \ i = 1, \dots, l_0$$

(7.23) $\Rightarrow$ $\forall x_s$ $1 \le s \le l_0$, $\lim_{i \to \infty} g_i(x_s) = \lim_{i \to \infty} f_{n_i}(x_s)$ exists.

$\exists N_s$ $\forall i, j \geq N_s$ $|g_i(x_s) - g_j(x_s)| < \varepsilon$

Let $N = \max(N_1, N_2, \dots N_s)$

$$\forall i, j \geq N \quad \forall s = 1, \dots, l_0 \quad |g_i(x_s) - g_j(x_s)| < \varepsilon \quad (*)$$

Let $x \in K$ be any pt, $x \in N_\delta(x_s)$ for some x_s.

$$|x - x_s| < \delta \Rightarrow |f_n(x) - f_n(x_s)| < \varepsilon$$

(Equicontinuity)

$$\Rightarrow |g_i(x) - g_i(x_s)| < \varepsilon$$

We Want uniform Caucy Criterion:

$\forall i,j \geq N$

$$|g_i(x) - g_j(x)| =$$

$$= |g_i(x) - g_i(x_s) + g_i(x_s) - g_j(x_s) + g_j(x_s) - g_j(x)|$$

$$\leq |g_i(x) - g_i(x_s)| +$$

$$|g_i(x_s) - g_j(x_s)| +$$ ⊛ p③

equicontinuity

$$|g_j(x_s) - g_j(x)|$$

$$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

Hence $\{g_i\}_{i=1}^{\infty}$ satisfies Uniform Cauchy Condition:

$\forall \varepsilon > 0\ \exists N\ \forall i,j \in N\ \forall x \in K\ |g_i(x) - g_j(x)| \leq 3\varepsilon$

$\Rightarrow \{g_i\}_{i=1}^{\infty} = \{f_{n_i}\}_{i=1}^{\infty}$ is Cauchy $\mathscr{C}(K)$.

$\Rightarrow$ " " is convergent in $\mathscr{C}(K)$.

<u>BIG PICTURE:</u> Chap 2. p 40: ① <u>Thm:</u> $\forall K \subseteq \mathbb{R}^n$: ($K$ is closed & bounded $\Leftrightarrow$ K compact)
(Heine-Borel)

② <u>Thm:</u> In Every metric space:
K compact $\Leftrightarrow$ Every infinite subset of K has a limit pt in K
$\Leftrightarrow$ Every sequence in K has a convergent subsequence in K (with limit in K)

ARZELA ASCOLI ③ <u>Thm:</u> Let K be a compact metric space. Let $\mathscr{C}(K, \mathbb{R})$ be the metric space of all continuous functions on K to $\mathbb{R}$, with the uniform metric, $\|f-g\|$. THEN:

A subset $\mathcal{F}$ $\subseteq \mathscr{C}(K, \mathbb{R})$ is compact $\Leftrightarrow$ $\mathcal{F}$ is closed, bounded & <u>equicontinuous</u>.