EQUICONTINUITY

Def A family of functions $F = \{ f_\alpha : E \to \mathbb{R} \mid \alpha \in A \}$ is called

- pointwise bounded if $\forall x \in E, \exists M_x \in \mathbb{R}$ s.t. $\forall f \in F, \ |f(x)| \le M_x$
- uniformly bounded if $\exists M \in \mathbb{R}$ s.t. $\forall f \in F \ \forall x \in E \quad |f(x)| \le M$
- Equicontinuous if $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall f \in F \ \forall x, y \in E$ $d(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon$

Obs F equicontinuous $\implies \forall f \in F$, f uniformly continuous

$$\not\Leftarrow$$

Ex $F = \{ f(x) = mx + b : [0,1] \to \mathbb{R} \mid m, b \in \mathbb{R} \}$

Given $M \in \mathbb{R}^+$, $F_M = \{ f(x) = mx + b : [0,1] \to \mathbb{R} \mid |m| \le M, b \in \mathbb{R} \}$

Given $M, B \in \mathbb{R}^+$ $F_{M,B} = \{ f(x) = mx + b : [0,1] \to \mathbb{R} \mid \begin{matrix} |m| \le M \\ |b| \le B \end{matrix} \}$

$\forall \underset{mx+b}{\underset{\parallel}{f}} \in F \ \forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{|m|+1}$

$d(x,y) < \delta \implies$

$$|f(x) - f(y)| \le |(mx+b) - (my+b)| = |m(y-x)| \le |m| \cdot |y-x|$$
$$< \delta \cdot |m| = \frac{|m|\,\varepsilon}{|m|+1} \le \varepsilon$$

(since larger $|m|$ require smaller δ)

$\mathcal{F}'$ is not equicontinuous, not uniformly and not pointwise bounded

(since b is not bounded)

F_M, $F_{M,B}$ are equicontinuous

M fixed "$\forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{M+1} > 0$" works for all m, $|m| \le M$

F_M is not uniformly / pointwise bdd $|b| \to +\infty$

$F_{M,B}$ is equicontinuous & uniformly bdd.

Recall Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence (Bolzano–Weierstrass)

Question (X, d) metric space $\mathcal{C}(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous}\}$

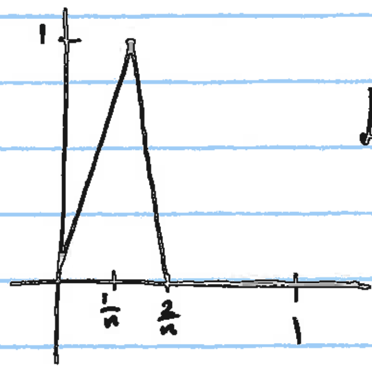
Does every bounded sequence in $\mathcal{C}(X)$ have a uniformly convergent subsequence?

Ans: NO in general See next page.

If X is compact, sequence is equicontinuous, (already bounded) then the sequence has a convergent subsequence. (Thm 7.25)

(Example 7.20 $f_n(x) = \sin nx$ How to read (after chap 11))

OUR Ex:



For $n \in \mathbb{N}$

let $f_n(x) = \begin{cases} nx & 0 \le x \le \frac{1}{n} \\ n\left(\frac{2}{n} - x\right) & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & x \ge 2/n \end{cases}$

$\{f_n(x)\}_{n=1}^{\infty}$ is uniformly bdd $\forall x \in [0,1]$ $\forall n \in \mathbb{N}$ $|f_n(x)| \le 1$

$f_n(x) \longrightarrow f \equiv 0$ pointwise

$\sup_{x \in [0,1]} \| f_n(x) - f(x) \| = 1$ $\quad \|f_n - f\| \not\to 0$

f_n doesn't converge to f uniformly.

$\{f_n\}$ can't be equicontinuous since

$\forall \varepsilon$, $\underbrace{\delta_n = \frac{\varepsilon}{n}}_{\text{best choice}}$ for f_n, due to the line segment of slope n.

As $n \to \infty$, one needs to choose δ_n smaller & we can't find $0 < \delta \le \underset{\downarrow \atop 0}{\delta_n}$ $\forall n$.

Every subsequence f_{n_k} of f_n has all of the properties above. Hence NO subsequence $f_{n_k} \to 0$ uniformly.

4 steps to get to Thm 7.25.

STEP I 7.23 **Thm** Let E be a countable set
$\forall n \; f_n : E \to \mathbb{R}$ be pointwise bdd.
Then f_n has a subsequence f_{n_k} s.t.
f_{n_k} converges $\forall x \in E$, as $k \to \infty$.

Proof: $\{x_i \mid i \in \mathbb{N}\} = E$ be an ordering of E

$\{f_n(x_1)\}_{n=1}^{\infty}$ a sequence of real #s,
it is bounded, since f_n is pointwise bdd.
Bolzano-Weierstrass tells us that $\{f_n(x_1)\}_{n=1}^{\infty}$ has a convergent subsequence:

$$f_{1,1}(x_1)\; f_{1,2}(x_1)\; f_{1,3}(x_1)\; f_{1,4}(x_1), f_{1,5}(x_1) \cdots$$

$$\lim_{k\to\infty} f_{1,k}(x_1) \text{ exists } = A_1$$

Now plug in x_2: $\underbrace{f_{1,1}(x_2), f_{1,2}(x_2), \ldots, f_{1,k}(x_2)}_{\text{bounded sequence}}, \ldots$

so it has a convergent subsequence:

$$f_{2,1}(x_2)\; f_{2,2}(x_2)\; f_{2,3}(x_2)\; f_{2,4}(x_2) \cdots \; f_{2,k}(x_2) \cdots$$

$$\lim_{k\to\infty} f_{2,k}(x_2) \text{ exists}, \qquad \underbrace{\lim_{k\to\infty} f_{2,k}(x_1) = A_1}_{\text{since } f_{2,k} \text{ is a subseq. of } f_{1,k}.}$$
$$= A_2$$

$S_1: \quad f_{1,1} \; f_{1,2} \; f_{13} \; f_{14} \; f_{15} \; \cdots \; f_{1k}$ converges to A_1 at x_1

$S_2: \quad f_{2,1} \; f_{22} \; f_{23} \; f_{24} \; f_{25} \quad f_{2k}$ conv. to A_2 at x_2

$S_3: \quad f_{3,1} \; f_{3,2} \; f_{33} \; f_{34} \; f_{35} \; \cdots \; f_{3k}$ conv. to A_3 at x_3

$\vdots$

- Each S_n is a subsequence of S_{n-1}
- $\{f_{n-1,k}(x_n)\}_{k=1}^{\infty}$ is a bounded sequence

& we take a convergent subsequence $\{f_{n,k}\}_{k=1}^{\infty}$

so that $\lim\limits_{k\to\infty} f_{n,k}(x_n)$ exists. $= A_n$.

- If one function g appears before h in row(n) then (i) g appears before h in all rows (ℓ), $\ell \le n$
  (ii) g appears before h in all rows (ℓ), $\ell > n$, only if both are still present, not removed.

Let $S: f_{11} \; f_{22} \; f_{33} \; f_{44} \; f_{55} \cdots f_{kk} \cdots$

S is a subsequence of S_n [th] except possibly first $n-1$ terms ($S_{\ell\ell}$ might be removed later)

$\Longrightarrow \lim\limits_{k\to\infty} f_{k,k}(x_n) = \lim\limits_{k\to\infty} f_{n,k}(x_n)$ exist. $= A_n$. THEN

$f_{k,k}$ is the subsequence we want.

STEP 2

Exc. 2.25 Every Compact K is seperable.

i.e. $\exists$ at most countable set $E \subseteq K$, $\bar{E} = K$.

Prof Given $n \in \mathbb{N}$

$$K \subseteq \bigcup_{p \in K} N_{\frac{1}{n}}(p)$$

$\exists$ finite subcover

$$K \subseteq \bigcup_{i=1}^{k_n} N_{\frac{1}{n}}(p_{n,i}) \qquad \forall n \quad \# p_{n,i} \text{ is finite}$$

$E = \{ p_{n,i} \mid n \in \mathbb{N} \ \& \ 1 \le i \le k_n \}$ is a

countable union of finite sets.

E is at most countable.

Let $x \in K$ be given

$\forall n \qquad x \in N_{\frac{1}{n}}(p_{n,i_n})$ for some $p_{n,i_n} \in E$.

$\forall n. \qquad d(x, p_{n,i_n}) < \frac{1}{n}$.

$$\lim_{n \to \infty} p_{n,i_n} = x. \qquad \{p_{n,i_n}\}_{n=1}^{\infty} \text{ is a sequence in } E.$$

$\forall x \in K$, $x \in \bar{E}$.