Chap II

Defn $f : \mathbb{N} \to A$ is called a sequence in A.

notation: $f(n) = x_n$; $x_1, x_2, x_3, \dots x_n, \dots$;

$\{x_n\}$; $\{x_n\}_{n=1}^{\infty}$

Caution:

Difference between $\underbrace{\{x_n\}}_{\text{a function.}}$ and $\underbrace{\{x_n \mid n \in \mathbb{N}\}}_{\text{set, actually the range.}}$?

Ex $(-1)^n$ $-1, 1, -1, 1, -1, \dots$ ← order is quite relevant

$\{(-1)^n \mid n \in \mathbb{N}\} = \{1, -1\}$ ← The set of values attained by the sequence.

WELL-ORDEREDNESS PROPERTY of $\mathbb{N}$

Every non-empty subset E of $\mathbb{N}$ must have a smallest element. Called $\min E$.

Obs: $\min E$ exist $\iff$ $\inf E$ exists & belongs to E

Well-Ord. Prop $\Longrightarrow$ Mathematical Induction

Thm: **M.I:** Let $P(n)$ be a statement depending on $n \in \mathbb{N}$.
If (i) $P(1)$ is true and
(ii) $\forall k \in \mathbb{N}$ $P(k) \Longrightarrow P(k+1)$, then

$P(n)$ is true for all natural numbers $n \in \mathbb{N}$.

Prop: LUB-property of $\mathbb{R} \Longrightarrow$ Well-orderedness of $\mathbb{N}$

Proof: Let $E \subseteq \mathbb{N}$, $E \neq \emptyset$.

E is bounded below by $1 = \min \mathbb{N}$.

LUB prop. $\Longrightarrow$ GLB-prop. in $\mathbb{R}$, $E \subseteq \mathbb{N} \subseteq \mathbb{R}$.

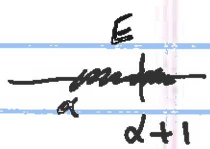
$\alpha = \inf E$ must exist in $\mathbb{R}$

$\alpha \geq 1$

If $\alpha \in E$, then $\alpha = \min E$. (

Suppose $\alpha \notin E$.

$\alpha + 1$ is not a lower bd for E



$\exists\, k \in E$ s.t. $\alpha \leq k < \alpha + 1$

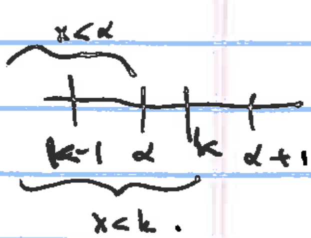
$\alpha \neq k$ since $\alpha \notin E$, $k \in E$

$\alpha < k < \alpha + 1$

$E \cap \{x \in \mathbb{R} \mid x < \alpha\} = \emptyset$ since $\alpha = \inf E$

$E \cap \{\underset{\mathbb{R}}{\overset{\uparrow}{x}} \mid k-1 < x < k\} \subseteq \mathbb{N} \cap \{x \mid k-1 < x < k\} = \emptyset$.

$k \in E \subseteq \mathbb{N}$.



$E \cap \left(\{x \in \mathbb{R} \mid x < k\} \right) = \emptyset$.

k is a lower bd for E, $k > \alpha = \inf E = \mathrm{glb}\, E$

Contradiction.

Hence $\alpha \in E$, $\alpha = \min E$. #

2.8 Thm: Every infinite subset of a countable set is countable.

Proof: Let. $E \subseteq A$, A countable, E infinite
WTS: E is countable.

A is countable $A \sim \mathbb{N}$.
$\exists\ g: \mathbb{N} \to A$ bijection.
$g(n) = x_n$ is a sequence of distinct elements in A, which exhausts A

$n_1 = \min \{ n \mid x_n \in E \}$ $\qquad n_1 \geq 1$

$n_2 = \min \{ n \mid x_n \in E,\ n \neq n_1 \}$ $\qquad n_2 > n_1$, $n_2 \geq n_1 + 1$; $\quad n_2 \geq 2$

;

$n_k = \min \underbrace{\{ n \mid x_n \in E,\ n \neq n_1, n_2, n_3, \ldots n_{k-1} \}}_{\text{never } \emptyset \text{ since } E \text{ is infinite.}}$ $\qquad n_k \geq n_{k-1} + 1 \geq k.$ $\quad n_k > n_{k-1}$

Let $f: \mathbb{N} \to E$ be

defined by $f(k) = x_{n_k}$

WTS (i) f is 1-1 (ii) f is onto.

f is 1-1.

Suppose $f(k_1) = f(k_2)$.

$$x_{n_{k_1}} = x_{n_{k_2}} \quad (\text{Def}^n \text{ of } f)$$

$$\Rightarrow g(n_{k_1}) = g(n_{k_2}) \quad (\text{Def}^n \text{ of } g)$$

$$\Rightarrow n_{k_1} = n_{k_2} \quad (\text{since } g \text{ is a bijection})$$

$$\Rightarrow k_1 = k_2 \quad \text{since} \quad n_1 < n_2 < n_3 < \dots < n_k < \dots$$

f is onto

$f: \mathbb{N} \to E$

Let $x \in E$ an arbitrary elt, $x \in E \subseteq A$.

$g: \mathbb{N} \to A$ onto

$\exists\, k$ s.t. $g(k) = x = x_k$

Suppose $x \notin f(\mathbb{N})$

$x_k = x \neq x_{n_1}, x_{n_2}, \dots x_{n_k}$ (via g: $\Rightarrow k \neq n_1, n_2, \dots, n_k$

$\exists$ contradiction in the following:

$k \in \underbrace{\{ n \mid x_n \in E,\ n \neq n_1, n_2, \dots n_k \}}_{E'}$.

$\Downarrow$ min of this set is $n_{k+1} \geq k+1$

$k \geq \min E' = n_{k+1} \geq k+1 > k$. (Contradiction)

Hence $f(\mathbb{N}) = E$.

f is a bijection, and $E \sim \mathbb{N}$. Countable.