April 27, 2018

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Uniform Convergence, Differentiation, Integration

7.16 **Thm** Let $\forall n$ $f_n : [a,b] \to \mathbb{R}$, bdd.

$\alpha : [a,b] \to \mathbb{R}$, $\alpha \uparrow$.

$\forall f_n \in R(\alpha)$

$f_n \to f$ uniformly

Then $f \in R(\alpha)$

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha$$

$$\parallel$$

$$\int_a^b \lim_{n\to\infty} f_n\,d\alpha$$

Proof: Let $\varepsilon_n = \| f_n - f \| = \sup_{x\in[a,b]} |f_n(x) - f(x)|$

$\varepsilon_n \to 0$ since $f_n \to f$ uniformly

$$\forall x \qquad |f_n(x) - f(x)| \le \varepsilon_n$$

$$f_n(x) - \varepsilon \le f(x) \le f_n(x) + \varepsilon_n$$

$$\underline{\int_a^b} f_n(x) - \varepsilon_n\,d\alpha \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f_n(x) + \varepsilon_n\,d\alpha.$$

$f_n \in R(\alpha)$ i.e.: $\overline{\int_a^b} = \underline{\int_a^b}$

$$\int_a^b (f_n(x) - \varepsilon_n)\,d\alpha \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le \int_a^b f_n(x) + \varepsilon_n\,d\alpha$$

difference $2\varepsilon_n \cdot (\alpha(b) - \alpha(a))$

$$\left| \underline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\,d\alpha \right| \leq 2\varepsilon_n(\alpha(b)-\alpha(a))$$

$\Big\downarrow$ let $n\to\infty$ so that $\varepsilon_n \to 0$.

$$0$$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \implies f \in \mathcal{R}(\alpha).$$

$$\int_a^b (f_n(x)-\varepsilon_n)\,d\alpha \leq \int_a^b f\,d\alpha \leq \int_a^b (f_n(x)+\varepsilon_n)\,d\alpha.$$

$$\int_a^b f\,d\alpha = \lim_{n\to\infty} \int_a^b f_n\,d\alpha.$$

__Corollary__ Let $f_n : [a,b] \to \mathbb{R}$ $\forall n$, $f_n \in \mathcal{R}(\alpha)$

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \quad \text{converging uniformly.}$$

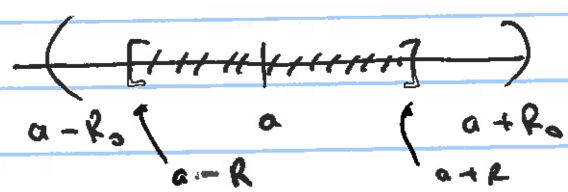
Then $$\int_a^b f\,d\alpha = \sum_{n=0}^{\infty} \int_a^b f_n(x)\,d\alpha$$

<u>Ex</u> $\sum_{n=0}^{\infty} a_n (x-a)^n$ $\qquad L_0 = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$

$$R_0 = \frac{1}{L_0}$$

$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ converges <u>pointwise</u>

on $J = (a - R_0, a + R_0)$



$0 < R < R_0$

$\sum_{n=1}^{\infty} a_n (R-a)^n$ convergent

Weierstrass-M-Test

WMT $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n (x-a)^n$ converges uniformly for $|x-a| \le R$.

$\Longrightarrow$ f is continuous on $|x-a| \le R$ $\forall R \le R_0$

$\Rightarrow$ f is continuous on $|x-a| < R_0$

Let $[c,d] \subseteq J$ $\qquad \int_c^d f(x)\,dx = \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} (x-a)^{n+1} \Big|_c^d \right)$

convergence is not uniform on $|x|<1$.

Ex $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x|<1$.

But unif convergence for $|x| \leq \varepsilon < 1$.

$$\sum_{n=0}^{\infty} \int x^n \, dx = \int \sum_{n=0}^{\infty} x^n \, dx = \int \frac{1}{1-x} \, dx$$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\ln|1-x| + C \quad \forall \ |x|<1.$$

Differentiation:

Repeat Ex. $f_n = \dfrac{\sin nx}{n} \longrightarrow f(x) \equiv 0$ uniformly

$f_n' = \cos nx \not\longrightarrow f'(x) \equiv 0$

So $\displaystyle\lim_{n\to\infty} \left(\frac{d}{dx} f_n(x) \right) \neq \frac{d}{dx} \lim_{n\to\infty} f_n(x)$

Crucial pt: You also need f_n' converge unif.

Thm 7.17 Let $\forall n$ $f_n : [a,b] \to \mathbb{R}$, be diff'ble.

$\{f_n(x_0)\}$ converge in $\mathbb{R}$ for some $x_0 \in [a,b]$

* $f_n'(x) \longrightarrow$ to a function uniformly

Then $f_n(x) \to f$ uniformly for some $f : [a,b] \to \mathbb{R}$
f is diff'ble and

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof: (Proof for diff'ble is in the book p 152-153)

We will give the proof for f_n' continuous.

f_n is continuously diff'ble, $\forall n$

Let $\forall n$ $g_n = f_n'$; Then g_n is continuous. $\forall n$.

$g_n \to g$ uniformly on $[a,b]$, for some g
continuous.
Thm 7.12

Define $f(x) = \int_{x_0}^{x} g(t)\,dt + A$
where $\lim_{n\to\infty} f_n(x_0) = A$,

so that $f(x_0) = A$.

We know $\begin{cases} \cdot\ f'(x) = g(x) & \text{(FTC)} \\ \cdot\ f_n'(x) = g_n \longrightarrow g(x) = f'(x) \end{cases}$

WTS $f_n \to f$ uniformly

$\forall x \in [a,b]$

Thm 7.16 today

$$f(x) - A = \int_{x_0}^{x} g(t)\,dt = \int_{x_0}^{x} \lim_{n\to\infty} g_n(t)\,dt = \lim_{n\to\infty} \int_{x_0}^{x} g_n(t)$$

$$= \lim_{n\to\infty}\left(f_n(x) - f_n(x_0)\right)$$

$$= \lim f_n(x) - A$$

$$\Longrightarrow \quad \lim_{n\to\infty} f_n(x) = f(x) \quad \text{Pointwise.}$$

We want uniform convergence.

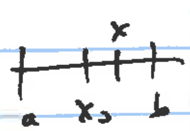
Let $\varepsilon > 0$. Choose N, ① $\forall n \geq N$ $|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{2}$, (Recall $f(x_0) = A = \lim_{n\to\infty} f_n(x_0)$), and

$\left(\begin{array}{l} g_n \to g \\ \text{uniformly} \end{array} \Longrightarrow\right)$ ② $\sup_{x\in[a,b]} |g_n(x) - g(x)| < \frac{\varepsilon}{2|b-a|}$.

$\forall x \in [a,b]$, $\forall n \geq N$

$$|f_n(x) - f(x)| = \left|\left(\int_{x_0}^{x} (f_n'(t) - f'(t))\,dt + (f_n(x_0) - f(x_0)\right)\right|$$

$$\leq \int_{x_0}^{x} |\underbrace{f_n'(t)}_{g_n} - \underbrace{f'(t)}_{g}|\,dt + |f_n(x_0) - f(x_0)|$$



$$\leq \int_{x_0}^{x} \frac{\varepsilon\,dt}{2|b-a|} + \frac{\varepsilon}{2} \leq \frac{|x_0 - x|}{|b-a|}\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon. \quad \#$$

$(|x_0 - x| \leq |b-a|)$