<u>Defn</u> Let $(X, d)$ be a metric space.

$$\mathcal{C}(X) = \{ f : X \to \mathbb{R} \mid f \text{ continuous } \& \; f \text{ bounded} \}$$

$\forall f \in \mathcal{C}(X)$, let $\|f\| = \sup_{x \in E} |f(x)| < \infty$

$$d_{\mathcal{C}}(f,g) = \|f - g\|$$

<u>Prop</u> $(\mathcal{C}(X), d_{\mathcal{C}})$ is a metric space.

<u>Proof:</u> $\|f\| = 0 \iff \sup |f(x)| = 0$
$\iff f(x) \equiv 0$

- $d_{\mathcal{C}}(f,g) = 0 \iff \|f-g\| = 0 \iff f - g \equiv 0 \iff f \equiv g$

- $d_{\mathcal{C}}(f,g) = \|f - g\| = \|g - f\| = d_{\mathcal{C}}(g,f)$

$\forall f, g \in \mathcal{C}(X) \quad \forall x \in X \quad |f(x) + g(x)| \le |f(x)| + |g(x)|$

$\forall x. \quad |f(x)| \le \|f\|$
$\forall x \quad |g(x)| \le \|g\|$

$\forall x \quad |f(x) + g(x)| \le \|f\| + \|g\|$

$\downarrow \sup$

$\|f + g\| \le \|f\| + \|g\|$

$\forall u, v \in \mathcal{C}(X)$

$d(u,w) = \|u - w\| = \|u - v + v - w\| \le \|u - v\| + \|v - w\| = d(u,v) + d(v,w)$

*** Thm 7.15 $(\mathscr{C}(X), d_{\mathscr{C}})$ is a complete metric space.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$

$f_n : X \to \mathbb{R}$

$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N$

$$d_{\mathscr{C}}(f_n, f_m) = \|f_n - f_m\| = \sup_{x \in X} |f_n(x) - f_m(x)| \leq \varepsilon$$

$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N \ \forall x \in X \quad |f_n(x) - f_m(x)| \leq \varepsilon$

Thm 7.8 $\Rightarrow \exists \ f : X \to \mathbb{R}$ s.t.

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \forall x \in X \quad |f_n(x) - f(x)| \leq \varepsilon$

Corollary 7.12 $\Rightarrow$ $f$ is continuous.

$\forall x \quad |f_N(x) - f(x)| \leq \varepsilon$

$\Rightarrow \forall x \ |f(x)| \leq |f_N(x)| + \varepsilon \ , \quad f_N \in \mathscr{C}(X)$

$\|f_N\| < \infty$

$\forall x \quad |f(x)| \leq \|f_N\| + \varepsilon$

$\|f\| < \infty$

$f \in \mathscr{C}(X)$.

$\forall \varepsilon > 0 \ \exists N \ \underset{\forall n \geq N}{\wedge} \ d_{\mathscr{C}}(f, f_n) = \|f - f_n\| = \sup_{x \in X} |f(x) - f_n(x)\| < \varepsilon$

Hence $f_n \to f$ w.r.t. $d_{\mathscr{C}}$.

**Ex** $\mathcal{C}(X)$, $X = [a,b]$, $\mathcal{L}^2$ metric

$$\langle f, g \rangle_2 = \int_a^b f(x) g(x)\, dx$$

$$\|f\|_2 = \left( \int_a^b f^2\, dx \right)^{1/2}$$

$$d_2(f,g) = \|f-g\|_2 \quad \text{distance function}$$

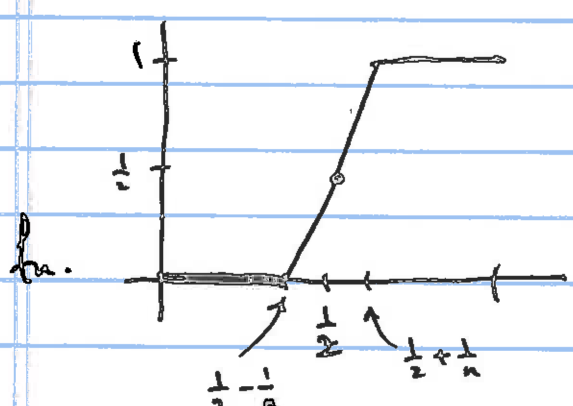
$$\|f\| = \sup_{x \in X} |f(x)| = A < \infty$$

$$\|f\|_2^2 = \int_a^b f^2(x)\, dx \le \int_a^b A^2\, dx = (b-a)A^2.$$

$$\|f\|_2 \le \sqrt{b-a}\, \|f\|$$

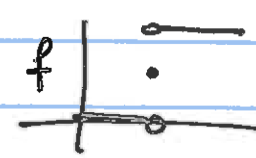
$$\text{Cauchy in } \|\cdot\| \implies \text{Cauchy in } \|\cdot\|_2$$

$$\not\Longleftarrow$$




$X = [0,1]$.

pointwise $f_n \to f \notin \mathcal{C}[0,1]$.




$$\|f_n - f_m\|_2 \le \frac{1}{\sqrt{6N}} \quad \forall n, m \ge N$$

$\{f_n\}$ is Cauchy in $\|\cdot\|_2$; but not Cauchy in $\|\cdot\|$.

*** $(\mathcal{C}(X), \|\cdot\|_2)$ is NOT complete, since $f \notin \mathcal{C}[0,1]$.

7.9 Thm: Let $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise.

Let $M_n = \sup_{x\in E} |f_n(x) - f(x)| \in [0,\infty]$

$f_n \to f$ uniformly $\iff \lim M_n = 0$.

Proof: $\forall x \quad |f_n(x) - f(x)| \leq \varepsilon \iff 0 \leq M_n < \varepsilon$

---

7.10 WEIERSTRASS-M-TEST.

Let $f_n : E \to \mathbb{R}$ fn. $\sup_{x\in E} |f_n(x)| = M_n$

$\sum_{n=1}^{\infty} M_n < \infty \implies \sum_{n=1}^{\infty} f_n(x)$ converges uniformly
($\&$ absolutely)(to $f$)

Ex $\sum_{n=1}^{\infty} \frac{e^x \sin nx}{n^2} \qquad 0 \leq x \leq 2\pi$

$0 \leq x \leq 2\pi \qquad |e^x| \leq e^{2\pi}$
$|\sin nx| \leq 1.$

$|M_n| \leq \frac{e^{2\pi}}{n^2}$

$\sum_{n=1}^{\infty} M_n \leq \sum_{n=1}^{\infty} \frac{e^{2\pi}}{n^2} = e^{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$

WMT $\Rightarrow \sum_{n=1}^{\infty} \frac{e^x \sin nx}{n^2}$ converges uniformly to $f$,
which is continuous
on $[0, 2\pi]$.

Remark: We can do this on any $[0, L]$, $L < \infty$; but not on $(0,\infty)$

## Proof of WMT

$$\sum_{n\geq 1}^{\infty} M_n < \infty \implies \text{Cauchy Criterion for series holds in } \mathbb{R}.$$

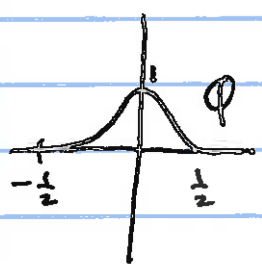
$$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N, \ m > n \quad \left| \sum_{k=n+1}^{m} M_k \right| < \varepsilon$$

$$\implies \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k < \varepsilon$$

CC for series of functions

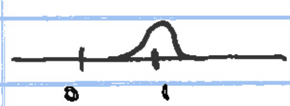
$$\implies \sum_{n=1}^{\infty} f_n(x) \quad \text{converges uniformly (Thm 7.8)}$$

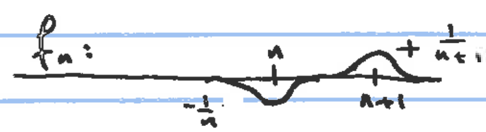
## Caution: Converse of WMT is false.




$$f_0 = \varphi(x-1)$$

$$f_n = -\frac{1}{n}\varphi(x-n) + \frac{1}{n+1}\varphi(x-(n+1))$$




$n \geq 1$

For $N \geq 1$
$$\sum_{n=0}^{N} f_n = \frac{1}{N+1}\varphi(x-(N+1)) \longrightarrow 0 \quad \text{uniformly}$$

since $\left\| \sum_{n=0}^{N} f_n \right\| = \frac{1}{N+1}$ for $N \geq 1$

but
$$\sum_{n\geq 0}^{\infty} M_n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$