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Ex

$$f_n(x) = \frac{\sin nx}{n}$$

$\forall x \in \mathbb{R}$ $\quad |f_n(x)| \le \frac{1}{n} \to 0$

Let $f(x) \equiv 0$; then $f_n(x) \to f(x)$ pointwise

$\forall \varepsilon > 0 \ \exists N > \frac{1}{\varepsilon} \ \forall n \ge N \ \forall x \in \mathbb{R};\ |f_n(x) - f(x)| = \left|\frac{\sin nx}{n} - 0\right| \le \frac{1}{n} < \varepsilon$

$f_n \to f(x)$ uniformly.

7.8 Thm Let $f_n : E \to \mathbb{R}$.

(i) f_n converges unif on E to a function $f : E \to \mathbb{R}$
i.e. $\forall \varepsilon > 0 \ \exists N \ \forall n \ge N \ \forall x \in E \quad |f_n(x) - f(x)| \le \varepsilon$

$\Longleftrightarrow$

(ii) Cauchy Criterion for uniform convergence CCUC
$\forall \varepsilon > 0 \ \exists N \ \forall m, n \ge N \ \forall x \in E \quad |f_m(x) - f_n(x)| \le \varepsilon$.

Proof (i) $\Rightarrow$ (ii):
$\forall \varepsilon > 0 \ \exists N \ \forall n \ge N \ \forall x \in E$
$|f_n(x) - f(x)| \le \frac{\varepsilon}{2}$.

If $m, n \ge N$ then $|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$

$\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

(ii) $\Rightarrow$ (i): $\{f_n\}$ is uniformly Cauchy (Given)

We do not have an f yet.

$\circledast$ Given $\forall \varepsilon > 0 \ \exists N \ \forall n,m \geq N \ \forall x \in E \quad |f_n(x) - f_m(x)| \leq \varepsilon$

Let $x \in E$, be fixed.

$\{f_n(x)\}$ is a sequence of real #s.

$\{f_n(x)\}$ is a Cauchy sequence of real #s.

$\mathbb{R}, \|\cdot\|$ Complete metric space $\Longrightarrow$ $f_n(x) \longrightarrow$ real # call it $f(x)$, for the chosen x.

we have $f(x) : E \longrightarrow \mathbb{R}$, $f_n(x) \rightarrow f(x)$ pointwise.

$\circledast$ $\forall \varepsilon > 0 \ \exists N \ \underset{\text{take } m \geq n}{\forall n,m \geq N} \ \forall x \in E \quad |f_n(x) - f_m(x)| \leq \varepsilon$

$$-\varepsilon \leq f_n(x) - f_m(x) \leq \varepsilon$$

Let $m \geq n$ fix, $\downarrow \infty$

$$f_m(x) - \varepsilon \leq f_n(x) \leq \varepsilon + f_m(x)$$

pointwise $\downarrow$ $\qquad \|$ $\qquad \downarrow$ pointwise

$$f(x) - \varepsilon \leq f_n(x) \leq \varepsilon + f(x)$$

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N : \left(|f_n(x) - f(x)| \leq \varepsilon \quad \text{true for all } x. \right)$

$f_n(x) \rightarrow f(x)$ uniformly.

VERY IMPORTANT

*** 7.11 Thm: Let $\forall n$, $f_n : E \to \mathbb{R}$, $f_n \to f$ uniformly
$f : E \to \mathbb{R}$.

Let $x \in E'$.

If $\lim\limits_{t \to x} f_n(t) = A_n \in \mathbb{R}$, then (i) $\lim\limits_{n \to \infty} A_n$ exists, and

(ii) $\lim\limits_{n \to \infty} A_n = \lim\limits_{t \to x} f(x)$

In other words:

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(x)$$

*** 7.12 Corollary: If $f_n : E \to \mathbb{R}$, $\forall n$ f_n continuous on E,
$f : E \to \mathbb{R}$,
and $f_n \to f$ uniformly

then f is continuous.

Proof of 7.11

(i) $\lim_{t\to x} f_n(t) = A_n \in \mathbb{R}$ given, $\forall n \in \mathbb{N}$

$f_n \to f$ uniformly (given)

$\{f_n\}$ uniformly Cauchy (Thm 7.8)

$$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N \ \forall t \in E \quad |f_n(t) - f_m(t)| \leq \varepsilon$$

$$-\varepsilon \leq f_n(t) - f_m(t) \leq \varepsilon$$

Let $t \to x$

$$-\varepsilon \leq A_n - A_m \leq \varepsilon$$

$\forall \varepsilon > 0 \ \exists N \ \forall n, m \geq N: \ |A_n - A_m| \leq \varepsilon$ i.e. Cauchy.

$\{A_n\}$ Cauchy $\implies \lim_{n\to\infty} A_n = A \in \mathbb{R}$.

(ii) $\lim_{n\to\infty} A_n \overset{\text{did it}}{=} A \overset{\text{wts}}{=} \lim_{t\to x} f(t)$

$$(*) \quad |f(t)-A| \le |f(t)-f_n(t)| + |f_n(t)-A_n| + |A_n - A|$$

Let $\varepsilon > 0$ be given.

Choose n s.t.

$f_n(t) \xrightarrow[\text{unif}]{} f(t) \implies$ (i) $|f(t)-f_n(t)| \le \frac{\varepsilon}{3}$ for all $t \in E$.

$\lim_{n\to\infty} A_n = A \implies$ (ii) $|A_n - A| \le \frac{\varepsilon}{3}$

For this given n, $\lim_{t\to x} f_n(t) = A_n$ which implies

$\exists V^{\subseteq E}_{\text{open}} \; \forall t \in (V \cap E) - \{x\} \quad |f_n(t) - A_n| \le \frac{\varepsilon}{3}$

Hence $(*)$ $|f(t) - A| < \varepsilon \quad \forall t \in (V \cap E) - \{x\}$,
for any given $\varepsilon > 0$

$$\lim_{n\to\infty} A_n = A = \lim_{t\to x} f(t)$$

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t) \quad \# \quad \text{(End of 7.11)}$$

Proof of Corollary 7.12 $f_n(t)$ continuous, given. $f_n \to f$ uniformly

$f(x) \overset{f_n\to f}{=} \lim_{n\to\infty} f_n(x) = \lim_{t\to x} f(t) \qquad f$ continuous.

since $f_n(t)$ continuous at x. $\qquad f_n \to f$