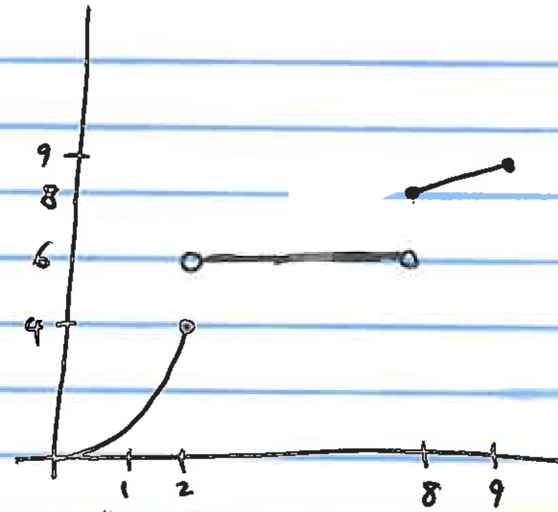
April 20, 2018

1

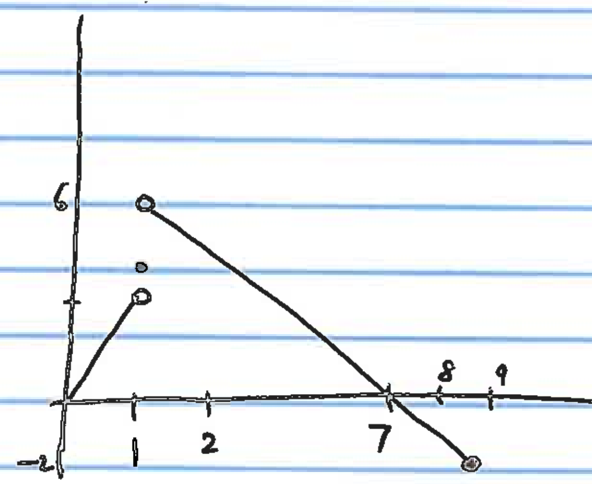
Chapter 6 to finish

Example

$$\alpha(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ 6 & 2 < x < 8 \\ x & 8 \le x \le 9 \end{cases}$$



$$f(x) = \begin{cases} 3x & 0 \le x < 1 \\ 4 & x = 1 \\ 7-x & 1 < x \le 9 \end{cases}$$



$$\int_0^9 f\,d\alpha = \int_0^1 3x(2x) + \int_1^2 (7-x)\cdot 2x + f(2)\cdot 2 + 0 + \int_8^9 (7-x)\cdot 1 d$$
$$+ 2f(8$$

$$= 2x^3\Big|_0^1 + 7x^2 - \frac{2}{3}x^3\Big|_1^2 + 2\cdot 5 + 7x - \frac{x^2}{2}\Big|_8^9 + 2\cdot(-1)$$

$$= 26\tfrac{5}{6}.$$

FUNDAMENTAL THMS OF CALCULUS

6.20 Thm. Let $f \in \mathcal{R}$ on $[a,b]$. For $a \le x \le b$, define

$$F(x) = \int_a^x f(t)\,dt. \text{ Then}$$

(i) F is continuous, and

(ii) If f is continuous at $x_0 \in [a,b]$, then F is diff'ble at x_0 & $F'(x_0) = f(x_0)$.

Proof: $f \in \mathcal{R} \implies f$ is bounded, $\exists M$ $|f(t)| \le M$ $\forall t$
take $M > 0$

If $a \le x < y \le b$ then

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \le M|y-x| \quad \text{by 6.12}$$

Uniform continuity $\forall \varepsilon > 0$ $\exists$ $\delta = \frac{\varepsilon}{M}$ $\forall x, y$ $|x-y| < \delta$

$$\implies |F(x) - F(y)| < \frac{\varepsilon}{M} \cdot M = \varepsilon$$

(ii) Assume f continuous at x_0.

$\forall \varepsilon > 0$ $\exists \delta > 0$ $|f(t) - f(x_0)| < \varepsilon \Leftarrow \begin{cases} |t - x_0| < \delta \\ \& \ t \in [a,b]. \end{cases}$

For s.t. $x_0 - \delta \le s < t \le x_0 + \delta$ and $s,t \in [a,b]$.

$$\left|\frac{F(t) - F(s)}{t-s} - f(x_0)\right| = \left|\left(\frac{1}{t-s}\int_s^t f(u)\,du\right) - f(x_0)\right|$$

$$= \left|\int_s^t \left(f(u) - f(x_0)\right) du\right| \frac{1}{t-s} \le \frac{1}{t-s}\int_s^t |f(u) - f(x_0)|\,du < \frac{1}{t-s}\int_s^t \varepsilon\,du = \varepsilon.$$

$\forall \varepsilon > 0 \ \exists \delta > 0 \quad |s-t| < \delta$

$$\Rightarrow \left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| < \varepsilon$$

$$F'(t_0) = \lim_{t \to x_0} \frac{F(t) - F(x_0)}{t - x_0} = f(x_0).$$

6.21 Thm: If $f \in \mathcal{R}$ on $[a,b]$ & if $\exists F: [a,b] \to \mathbb{R}$ s.t. $F' = f$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof: Let $\varepsilon > 0$ be given

Choose $P = \{x_0, x_1, \dots x_n\}$ s.t. $U(P,f) - L(P,f) < \varepsilon$

MVT $\Rightarrow \exists t_i \in [x_{i-1}, x_i]$ s.t. $F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i$

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n} F'(t_i)\Delta x_i = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) = F(b) - F(a).$$

Thm 6.7(c): $\underbrace{L(P,f) \le \sum f(t_i)\Delta x_i \le U(P,f)}_{\varepsilon\text{-apart at most}}$

$$\Rightarrow \underbrace{L(P,f) \le F(b) - F(a) \le U(P,f)}_{\text{at most } \varepsilon \text{ apart}}$$

$$\Rightarrow \int_a^b f\,dx = F(b) - F(a)$$

Chap VII

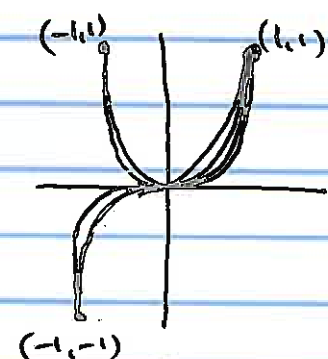
HW to read Examples 7.2–7.6 (In essence, similar to those below.)

7.1 Defn Let $f_n : E \to \mathbb{R}$ be defined $\forall n$.
$\{f_n(x)\}$ is called a sequence of functions.
Let $f : E \to \mathbb{R}$ be given.
If $\underset{\text{fixed}}{\forall x \in E}$ $f_n(x) \to f(x)$, then

$f_n(x)$ is called to converge pointwise to $f(x)$.

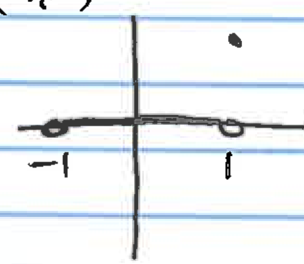
i.e. $\forall x \in E \; \forall \varepsilon > 0 \; \exists N = N(x, \varepsilon) \; \forall n \geq N \quad |f_n(x) - f(x)| < \varepsilon$

Ex 1 $f_n(x) = x^n : [-1, 1] \to \mathbb{R}$



If $|x| < 1$, $\quad x^n \to 0$
$x = 1$ $\quad 1^n \to 1$.
$x = -1$ $\quad (-1)^n \not\to$

Let $f(x) = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$



$\forall n$ $f_n(x)$ is continuous.

$f_n(x) \to f(x)$ pointwise on $(-1, 1]$.

$$\lim_{x \to 1} \lim_{n \to \infty} f_n(x) = 0$$
$$\nparallel$$
$$\lim_{n \to \infty} \lim_{x \to 1} f_n(x) = 1$$

$f(x)$ is not continuous at 1.

Ex 2

$$S_{n,m} = \begin{cases} 0 & \text{if } n \geq m \\ 1 & \text{if } n < m \end{cases}$$

$$\begin{array}{c|ccccccc}
 & \xrightarrow{m} & & & & & & \\
\hline
n \downarrow & 0 & 1 & 1 & 1 & 1 & & \to 1 \\
 & 0 & 0 & 1 & 1 & 1 & & \to 1 \\
 & 0 & 0 & 0 & 1 & 1 & & \to 1 \\
 & 0 & 0 & 0 & 0 & 1 & & \\
 & 0 & 0 & 0 & 0 & 0 & & \\
 & \downarrow & \downarrow & \downarrow & & & & \\
 & 0 & 0 & 0 & & & &
\end{array}$$

$$\lim_{n\to\infty} \lim_{m\to\infty} S_{n,m} = \lim_{n\to\infty} 1 = 1$$

$$\nparallel$$

$$\lim_{m\to\infty} \lim_{n\to\infty} S_{n,m} = \lim_{m\to\infty} 0 = 0$$

Ex 3

$$\sum_{n=0}^{\infty} \frac{x}{(1-x)^n} = \begin{cases} 0 & \text{if } x=0 \\ x \cdot \frac{1}{x} = 1 & \text{if } 0<x<1 \end{cases}$$

$0 \leq x < 1$

$$\left(\sum_{n=0}^{\infty} \frac{1}{(1-x)^n} = \frac{1}{1-(1-x)} = \frac{1}{x} \right)$$

$1-x < 1$

Partial sums: $\sum_{n=0}^{M} \frac{x}{(1-x)^n}$ are continuous on $[0,1)$; But $\sum_{n=0}^{\infty} \frac{x}{(1-x)^n}$ is not continuous at 0.

Ex 4

$$f_n(x) = \frac{\sin nx}{n} \longrightarrow 0 \quad \forall x \quad \text{pointwise}$$

(actually uniformly, later on)

$$\left| \frac{\sin nx}{n} - 0 \right| \le \frac{1}{n}. \qquad |\sin nx| \le 1$$

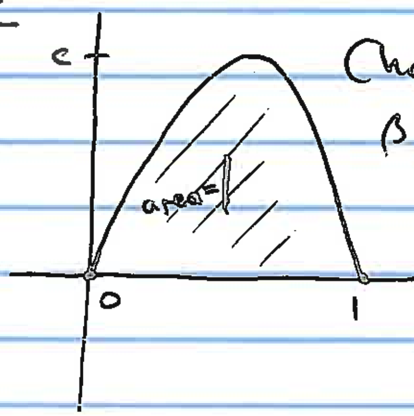
$$f_n(x) \to f(x) \equiv 0$$

But $f_n'(x) \not\to f'(x)$

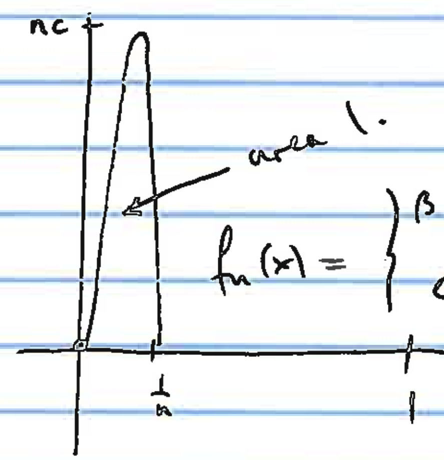
$$f_n'(x) = \cos nx. \not\longrightarrow f'(x) = 0$$

if $x \neq 2k\pi$.

Ex 5



Choose a $\beta(x)$



$$f_n(x) = \begin{cases} \beta(nx)\cdot n & \forall 0<x\le\frac{1}{n} \\ 0 & 0 \end{cases}$$

$$f_n(x) \longrightarrow f(x) \equiv 0, \text{ since}$$

$\forall x \in (0,1]$ given, fixed $\quad \exists n \in \mathbb{N}, \; n > \frac{1}{x}$ i.e. $\frac{1}{n} < x$

$\forall m \ge n \quad f_m(x) = 0$

$$\lim_{n\to\infty} \underbrace{\int_0^1 f_n(x)\,dx}_{1} = \lim_{n\to\infty} 1 = 1 \neq 0 = \int_0^1 f(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

7.7 **Defn** Let $f_n : E \to \mathbb{R}$ $\forall n \in \mathbb{N}$ be given

pointwise • $f_n(x)$ is said to converge to a function $f: E \to \mathbb{R}$.
if for each x, $f_n(x) \to f(x)$

$\forall x \in E \ \forall \varepsilon > 0 \ \exists N \ \forall n \geq N \quad |f_n(x) - f(x)| \leq \varepsilon$

uniformly • $f_n(x)$ is said to converge uniformly to a a function $f: E \to \mathbb{R}$ if

$$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \forall x \in E \quad |f_n(x) - f(x)| \leq \varepsilon$$

Ex 1 revisited

$$f_n(x) = x^n : [0,1] \to \mathbb{R}$$

$$f_n(x) \to f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

not uniform convergence
but pointwise convergence

WTS $\exists \varepsilon > 0 \ \forall N \ \exists n \geq N \ \exists x \in E \quad |f_n(x) - f(x)| > \varepsilon$

Let $\varepsilon = \frac{1}{2}$ (want $|x^n - 0| \geq \frac{1}{2}$)

since $\sqrt[n]{1/2} \nearrow 1$ $\quad \sqrt[n]{1/2} < 1$

$\exists \varepsilon = \frac{1}{2}$, $\forall N$ choose $n = N$ $\exists x$ $\sqrt[n]{1/2} < x < 1$

$$\frac{1}{2} < x^n < 1$$

$|x^n - 0| > 1/2$. $\quad (x < 1 \Rightarrow f(x) = 0)$