April 18, 2018

1

6.12 HW to read p128 Basic Properties.

6.13 Thm: Let $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$ on $[a,b]$. Then

i) $fg \in \mathcal{R}(\alpha)$

ii) $|f| \in \mathcal{R}(\alpha)$, $\left| \int_a^b f\,d\alpha \right| \leq \int_a^b |f|\,d\alpha$.

Proof: (i) $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha) \Rightarrow f+g,\ f-g \in \mathcal{R}(\alpha)$

Thm 6.11 $\Rightarrow f^2, (f+g)^2, (f-g)^2 \in \mathcal{R}(\alpha)$

$4fg = (f+g)^2 - (f-g)^2 \in \mathcal{R}(\alpha)$

(ii) $\phi(t) = |t|$ is continuous $|f| \in \mathcal{R}(\alpha)$.

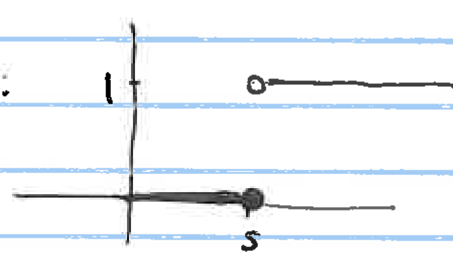
$\int_a^b f\,d\alpha$ Let $c = \operatorname{sgn}\left(\int_a^b f\,d\alpha\right) = 0$ or 1 or -1

$$\left|\int_a^b f\,d\alpha\right| = c\int_a^b f\,d\alpha = \int_a^b cf\,d\alpha \leq \int_a^b |f|\,d\alpha.$$

$cf \leq |f|$

6.14 Def $I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$

$I(x-s)$:



6.15 **Thm:** Let $a < s < b$, $f : [a,b] \to \mathbb{R}$ bdd

f continuous at s

$\alpha = I(x-s)$

then $\displaystyle\int_a^b f\,d\alpha = f(s)$

Proof: Let $\varepsilon > 0$ be given $\forall \varepsilon > 0 \; \exists \delta > 0$

$$|t-s| < 2\delta \implies |f(t) - f(s)| < \varepsilon$$

$$f(s) - \varepsilon < f(t) < f(s) + \varepsilon \quad \forall t \in (s-2\delta, s+2\delta)$$

$$t \in \underbrace{[s-\delta, s+\delta]}_{J_\delta}$$

$$f(s) - \varepsilon \le \underbrace{\inf_{J_\delta} f(t)}_{m_2} \le \underbrace{\sup_{J_\delta} f(t)}_{M_2} \le f(s) + \varepsilon$$

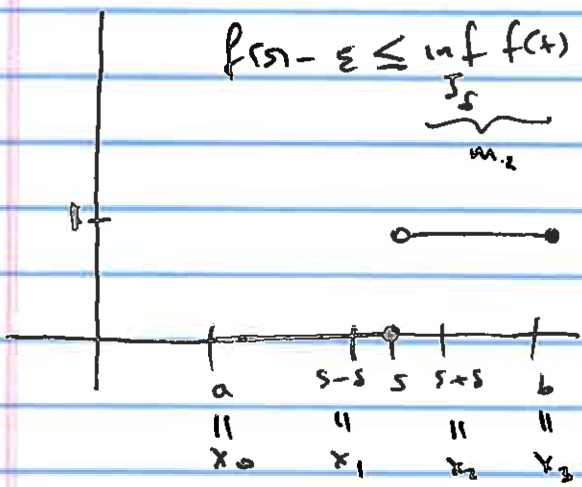
Let $P = \{ x_0 = a,$
$x_1 = s - \delta,$
$x_2 = s + \delta,$
$x_3 = b \}$.

$I(x-s) = \alpha$

$\Delta\alpha_1 = 0$

$\Delta\alpha_2 = 1$

$\Delta\alpha_3 = 0$



$$L(P,f,\alpha) = m_2 \underbrace{\Delta\alpha_2}_{1} \le M_2 \underbrace{\Delta\alpha_2}_{1} = U(P,f,\alpha)$$

$$f(s) - \varepsilon \le L(P,f,\alpha) = m_2 \le M_2 = U(P,f,\alpha) \le f(s) + \varepsilon \quad (*)$$

$$|U(P,f,\alpha) - L(P,f,\alpha)| \le 2\varepsilon$$

As $\varepsilon \to 0$

$$\implies \int_a^b f\,d\alpha \text{ exists } = f(s) \quad (*)$$

6.16 Thm Let $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$

Let s_n be a sequence in (a,b),
s_n be distinct i.e. $s_n \neq s_m$ if $n \neq m$.

$f : [a,b] \to \mathbb{R}$ continuous

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$$

Then $f \in \mathcal{R}(\alpha)$, $\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} f(s_n) c_n$.

Proof HW to read. (proof is not in the test)

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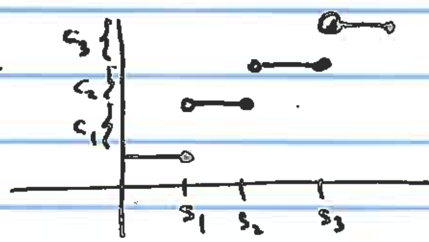
① Finite Case



α.

By Thm 6.12(e)

$$\int_a^b f \, d\alpha = f(s_1) \cdot c_1 + f(s_2) c_2 + f(s_3) c_3$$

② Ex let $p : \mathbb{N} \to \mathbb{Q} \cap (0,1)$ be a bijection. $p(n) = p_n$.

Let $c_n > 0$, $C = \sum_{n=0}^{\infty} c_n < \infty$

Let $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - p_n)$ (i) $\alpha(x) \in [0, C]$. bdd

(ii) $\alpha \nearrow$.

need $c_n \geq 0 \implies$ (iii) α is not continuous on p_n.

i.e. on $\mathbb{Q} \cap (0,1)$.

(iv) α continuous at each irrational number.

$\forall f$ continuous on $[a,b]$

$$\alpha = \sum_{n=1}^{\infty} c_n I(x-p_n), \qquad \int_a^b f\,d\alpha = \sum_{n=1}^{\infty} f(s_n)c_n.$$

$$\left| \sum_{n=1}^{\infty} f(s_n)c_n \right| \leq MC, \qquad M = \max|f(x)|$$

$$C = \sum_{n=1}^{\infty} c_n$$

Remark: Read 4.30 & 4.31 about this example.

- One can't have an increasing function with discontinuities at each irrational number. $\Leftarrow$ Thm 4.30
- There are functions which are discontinuous on both rationals & irrationals:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q}. \end{cases}$$

Google Check Popcorn function Thomae's function

6.17 Thm: Let $f : [a,b] \to \mathbb{R}$ bdd.

$\alpha : [a,b] \to \mathbb{R}$, $\alpha \uparrow$, α diff'ble on $[a,b]$

$\alpha' \in \mathcal{R}$.

Then $f \in \mathcal{R}(\alpha) \iff f\alpha' \in \mathcal{R}$ ← means standard Riemann integr.

In both cases: $\int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$

6.17 **Proof** Let $\varepsilon > 0$

$\exists P = \{x_0, x_1, \dots x_n\}$ of $[a,b]$

standard Riemann sums

$$U(P,\alpha') - L(P,\alpha') < \varepsilon$$

$\underline{\text{MVT}} \Rightarrow \exists\, t_i \in [x_{i-1}, x_i] \quad \alpha'(t_i) \cdot \Delta x_i = \Delta \alpha_i$

fixed

$\forall s_i \in [x_{i-1}, x_i]$ $\quad \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \leq \sum (M_i - m_i) \Delta x_i < \varepsilon$

free

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} \alpha'$$

$$m_i = \inf_{x_{i-1} \leq x \leq x_i} \alpha'$$

Let $M = \sup |f(x)|$

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum f(s_i)\alpha'(t_i)\Delta x_i$$

in general: $s_i \neq t_i$

changed $\alpha'(t_i)$ to $\alpha'(s_i)$

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i \right| \leq M\varepsilon$$

$$\sum f(s_i)\Delta\alpha_i \leq \sum f(s_i)\alpha'(s_i)\Delta x_i + M\varepsilon$$

$\downarrow \sup_{s_i \in [x_{i-1}, x_i]}$

$$\sum f(s_i)\Delta\alpha_i \leq \underbrace{U(P, f\alpha')}_{\text{fixed}} + M\varepsilon$$

$\downarrow \sup_{s_i \in [x_{i-1}, x_i]}$

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon$$

Similarly

$$U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon$$

$$|U(P, f\alpha') - U(P, f, \alpha)| \leq M\varepsilon$$

Similarly

$$\left| \overline{\int_a^b} f\alpha' dx - \overline{\int_a^b} f\,d\alpha \right| \leq M\varepsilon \quad \forall \varepsilon$$

$$\overline{\int_a^b} f\alpha' dx = \overline{\int_a^b} f\,d\alpha.$$

Similarly

$$\underline{\int_a^b} f\alpha' dx = \underline{\int_a^b} f\,d\alpha.$$

$$\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha \iff f \in \mathcal{R}(\alpha) \iff f\alpha' \in \mathcal{R} \iff \underline{\int_a^b} f\alpha' dx = \overline{\int_a^b} f\alpha' dx$$

In All cases: $\displaystyle\int_a^b f\alpha' dx = \int_a^b f\,d\alpha$