6.10 Thm

Let $f, \alpha : [a,b] \to \mathbb{R}$, $\alpha \uparrow$, f bounded.

Let $E_f = \{ x \in [a,b] \mid f$ is NOT continuous at $x \}$.

If (i) E_f is a finite set, and
(ii) α is continuous on E_f

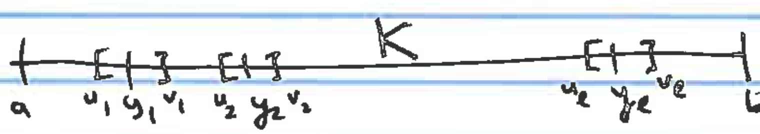
Then $f \in R(\alpha)$

Proof: Let $E_f = \{ y_1, y_2, \ldots y_\ell \}$,
$y_1 < y_2 < y_3 < \cdots < y_\ell$

$\forall i$ f is discontinuous at y_i

$\forall i$ α is continuous at y_i

Let $\varepsilon > 0$ be given.



Choose $[u_i, v_i]$ s.t.

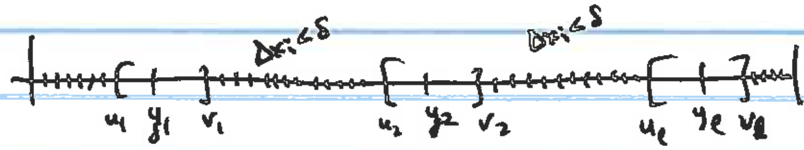
1) all $[u_i, v_i]$ are pairwise disjoint
2) $y_i \in (u_i, v_i)$ unless $y_1 = a$ OR $y_\ell = b$
Take ($v_1 > a$, $u_1 = a$ $\quad$ $u_\ell < b$, $v_\ell = b$)

3) α cont at $y_i \Rightarrow \alpha(v_j) - \alpha(u_j) < \varepsilon$

Let $K = [a,b] - \bigcup_{j=1}^{\ell} (u_j, v_j)$. closed $\subseteq [a,b]$
K is compact

$$E_f \cap K = \phi.$$

f cont on $K \Longrightarrow f$ is unif. cont on K.



(**) For given ε, $\exists\, \delta > 0 \ \wedge\ |x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$
$\forall x, y \in K$.

Construct a partition $P = \{x_0, x_1, \ldots x_n\}$ of $[a,b]$ s.t.

(i) All $u_i, v_j \in P$

(ii) $P \cap (u_j, v_j) = \phi \ \forall j$

(iii) $\Delta x_i < \delta$ if $(x_{i-1}, x_i) \cap \bigcup_{j=1}^{\ell} (u_j, v_j) = \phi$.

Define i to be $\begin{cases} \text{Type I}: & (x_{i-1}, x_i) \cap \bigcup_{j=1}^{\ell} (u_j, v_j) = \phi \\ \text{Type II}: & [x_{i-1}, x_i] = [u_j, v_j] \text{ for some } j. \end{cases}$

<u>Type I</u> $\quad \Delta x_i < \delta \Rightarrow M_i - m_i \le \varepsilon$

<u>Type II</u> $\quad M_i - m_i \le 2M \qquad M = \sup_{[a,b]} |f|$.

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \sum_{\mathrm{I}} (M_i - m_i)\Delta\alpha_i + \sum_{\mathrm{II}} (M_i - m_i)\Delta\alpha_i$$

$$\le \varepsilon \underbrace{\sum_{\mathrm{I}} \Delta\alpha_i} + \underbrace{\sum 2M\,(\alpha(v_j) - \alpha(u_j))}_{\text{via } (**)}$$

$$\le \varepsilon(\alpha(b) - \alpha(a)) + 2M \cdot \varepsilon$$

$$= \varepsilon\,\underbrace{(2M + \alpha(b) - \alpha(a))}_{\text{fixed \#}} \qquad \#$$

6.11 **Thm:** Let $\alpha : [a,b] \to \mathbb{R}$ $\quad \alpha \uparrow$

$f : [a,b] \to [m,M]$ $\quad f$ bdd $\quad f \in \mathcal{R}(\alpha)$

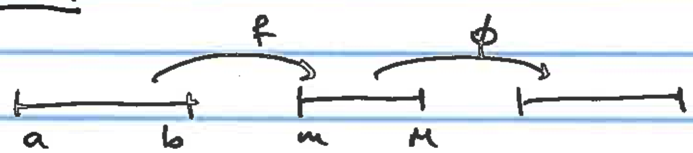
$\phi : [m,M] \to \mathbb{R}$ $\quad$ continuous

Let $h = \phi \circ f : [a,b] \to \mathbb{R}$

Then $h \in \mathcal{R}(\alpha)$.

Ex: $f \in \mathcal{R}(\alpha) \implies f^2 \in \mathcal{R}(\alpha) \quad (\phi(t) = t^2)$

Proof Let $\varepsilon > 0$



$[m,M]$ compact, ϕ continuous $\implies \phi$ unif continuous.

(*) Given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\delta < \varepsilon$, $\forall x, y \in [m,M]$, $(|\phi(x) - \phi(y)| < \varepsilon$ $\Leftarrow |x-y|<$

$f \in \mathcal{R}(\alpha)$ $\exists P = \{x_0, x_1, \dots x_n\}$ of $[a,b]$ s.t.

$$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

Let $M_i = \sup\limits_{x_{i-1} \le x \le x_i} f \qquad M_i^* = \sup\limits_{x_{i-1} \le x \le x_i} h$

$m_i = \inf\limits_{x_{i-1} \le x \le x_i} f \qquad m_i^* = \inf\limits_{x_{i-1} \le x \le x_i} h$

$I = \{1, 2, 3, \dots n\} = A \cup B$, $A \cap B = \phi$, where

Specify $i \in A$ if $M_i - m_i < \delta$

Specify $i \in B$ if $M_i - m_i \geq \delta$

If $i \in A$: $M_i - m_i < \delta \implies M_i^* - m_i^* < \varepsilon$ by (*) p(3)

Let $C = \max |\phi(t)|$

If $i \in B$: $M_i^* - m_i^* \le 2C$

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \le \sum_{i \in A \cup B} (M_i - m_i)\Delta\alpha_i$$

$$\le U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

Divide by δ:

$$\sum_{i \in B} \Delta\alpha_i < \delta$$

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i$$

$$\le \varepsilon \sum_{i \in A} \Delta\alpha_i + 2C \sum_{i \in B} \Delta\alpha_i$$

$$= \varepsilon(\alpha(b) - \alpha(a)) + 2C \cdot \delta < \varepsilon(\alpha(b) - \alpha(a) + 2C).$$

(use $\delta < \varepsilon$)

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