6.5 **Prop** Let $f, \alpha : [a,b] \to \mathbb{R}$, $\alpha \uparrow$, f bdd.

Then $$\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha.$$

Proof Let P_1, P_2 be partitions of $[a,b]$, set $P = P_1 \cup P_2$.

$$L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_2, f, \alpha)$$

$$\underline{\int_a^b} f\,d\alpha = \sup_{P_1} L(P_1, f, \alpha) \le U(P_2, f, \alpha) \quad \forall P_2$$

since P_1 & P_2 are independent

$$\underline{\int_a^b} f\,d\alpha \le \inf_{P_2} U(P_2, f, \alpha) = \overline{\int_a^b} f\,d\alpha$$

$$\underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha.$$

6.6 Thm Let $f, \alpha : [a,b] \to \mathbb{R}$, $\alpha \uparrow$, f bdd.

$$f \in \mathcal{R}(\alpha) \iff \begin{cases} \forall \varepsilon > 0 \ \exists \text{ a partition } P \text{ of } [a,b] \\ \text{s.t. } U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon. \end{cases}$$

Proof ($\Leftarrow$:)

Assume $\forall \varepsilon > 0 \ \exists P$ s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$$L(P,f,\alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P,f,\alpha)$$

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

$$\Longrightarrow \quad \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon \quad (\forall \varepsilon > 0)$$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha.$$

$$f \in \mathcal{R}(\alpha).$$

($\Leftarrow$:) Assume $f \in \mathcal{R}(\alpha)$

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha. = \overset{\text{Call it}}{A}$$

$$\sup_P L(P,f,\alpha) = \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha)$$

$$\sup_P L(P,f,\alpha) = A = \inf_P U(P,f,\alpha)$$

Let $\varepsilon > 0$ be given

$\exists P_1$ s.t. $U(P_1, f, \alpha) < A + \frac{\varepsilon}{2}$ (Def^n of inf)

$\exists P_2$ s.t. $L(P_2, f, \alpha) > A - \frac{\varepsilon}{2}$ (Def of sup)

If we take $P = P_1 \cup P_2$ then

$$\underbrace{A - \tfrac{\varepsilon}{2} < L(P_2, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_1, f, \alpha) < A + \tfrac{\varepsilon}{2}}_{\varepsilon\text{-apart}}$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

6.7 Consequences Let ⊛ denote $\left\{ \begin{array}{l} \forall \varepsilon > 0 \ \exists P \text{ s.t.} \\ U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \end{array} \right.$

(a) $\left. \begin{array}{l} \text{⊛ holds for } P \\ P \subseteq P^* \end{array} \right\} \Longrightarrow$ ⊛ holds for P^*

(b) ⊛ holds for $P = \{x_0, x_1, \dots x_n\}$

$\Longrightarrow \forall s_i, t_i \in [x_{i-1}, x_i]$, $\displaystyle\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon$

(why?

$U - L = \sum (M_i - m_i) \Delta\alpha_i < \varepsilon$

$m_i \le f(s_i) \,\&\, f(t_i) \le M_i \Longrightarrow |f(s_i) - f(t_i)| \le M_i - m_i$.)

(c) $f \in \mathcal{R}(\alpha)$ & b holds then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon$$

Why: $m_i \le f(t_i) \le M_i \Longrightarrow L(P, f, \alpha) \le \sum f(t_i) \Delta\alpha_i \le U(P, f, \alpha)$

$L(P, f, \alpha) \le \int_a^b f \, d\alpha \le U(P, f, \alpha)$

6.8 Thm $f, \alpha : [a,b] \longrightarrow \mathbb{R}$, $\alpha \uparrow$, f continuous

$\Longrightarrow f \in \mathcal{R}(\alpha)$.

Proof Let $\varepsilon > 0$ be given.

Choose $\eta > 0$ s.t. $|\alpha(b) - \alpha(a)| \eta < \varepsilon$

f continuous on $[a,b]$ which is compact

$\Rightarrow f$ is uniformly continuous on $[a,b]$, α bounded.

(*) $\exists \delta > 0$ s.t. $\forall x, t \in [a,b]$ $|x-t| < \delta \Longrightarrow |f(x) - f(t)| < \eta$.

Choose any partition P of $[a,b]$ s.t. $\Delta x_i < \delta$.

(e.g. $x_i = a + i\frac{\delta}{2}$ or divide $[a,b]$ into n equal pieces where $n = \left[\!\left[\frac{2(b-a)}{\delta} + 1 \right]\!\right]$

On $[x_{i-1}, x_i]$ $\quad m_i = \inf\limits_{x_{i-1} \le x \le x_i} f$

$M_i = \sup\limits_{x_{i-1} \le x \le x_i} f$

Since $\Delta x_i < \delta$ (*) $\Longrightarrow |M_i - m_i| \le \eta$.

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i \le \sum_{i=1}^{n} \eta \Delta\alpha_i$$

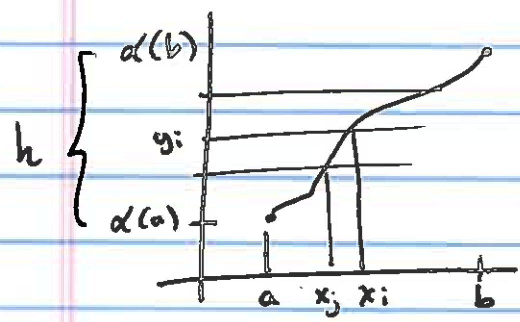
$$= \eta \sum_{i=1}^{n} \Delta\alpha_i = (\alpha(b) - \alpha(a)) \cdot \eta < \varepsilon$$

By Thm 6.6 $f \in \mathcal{R}(\alpha)$

6.9 Thm

$$\left.\begin{array}{l} f, \alpha : [a,b] \to \mathbb{R} \\ \alpha \uparrow, \text{ continuous} \\ f \uparrow \end{array}\right\} \Longrightarrow f \in \mathcal{R}(\alpha)$$

Proof: Given $n \in \mathbb{N}$



Divide interval $[\alpha(a), \alpha(b)]$ into n equal pieces

$\alpha(a) = y_0 \le y_1 \le \dots \le y_n = \alpha(b)$.

$$\Delta y_i = \frac{\alpha(b) - \alpha(a)}{n} = \frac{h}{n}$$

By Intermediate Value Thm.

$\exists\, x_i \in [a,b], \quad \alpha(x_i) = y_i$.

Let $P_n = \{x_0, \dots x_n\}$. Observe that $\Delta y_i = \Delta \alpha_i = \frac{h}{n}$

$f \uparrow \Longrightarrow M_i = f(x_i)$ max of f on $[x_{i-1}, x_i]$

$m_i = f(x_{i-1})$ min of f " "

$$U(P_n, f, \alpha) - L(P_n, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta \alpha_i$$

$$= \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \cdot \underbrace{\frac{h}{n}}_{\text{fixed}} = \frac{h}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{h}{n}\big(f(b) - f(a)\big) = \frac{(\alpha(b) - \alpha(a))(f(b) - f(a))}{n} < \varepsilon$$

if we take $n > \dfrac{(\alpha(b) - \alpha(a))(f(b) - f(a))}{\varepsilon}$.

Then use Thm 6.6. to obtain $f \in \mathcal{R}(\alpha)$.