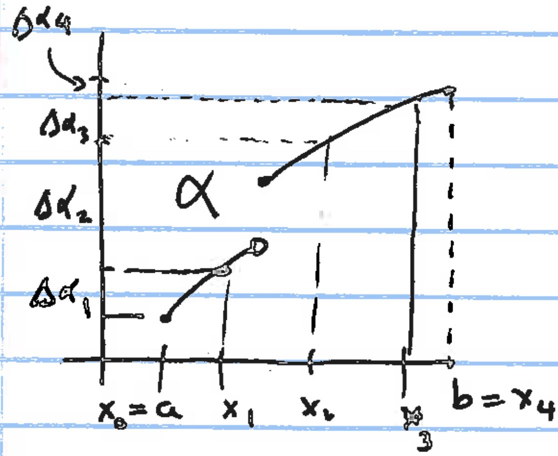
Chap VI: Riemann-Stieltjes Integral

HW read p120-121 for Riemann integral ($\sim$ as in Calc II.)

Let $[a,b] \subseteq \mathbb{R}$,

$\alpha : [a,b] \to \mathbb{R}$, $\alpha \uparrow$: $x \geq y \Rightarrow \alpha(x) \geq \alpha(y)$

$f : [a,b] \to \mathbb{R}$, f bounded



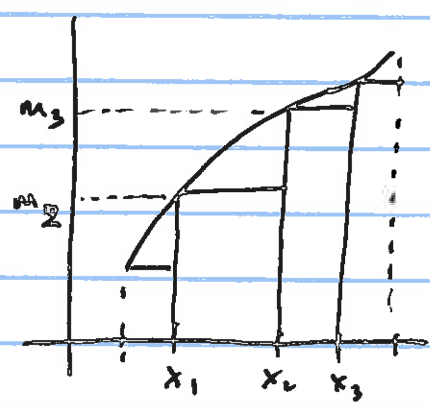
A partition $P = \{x_0, x_1, \ldots x_n\}$ of $[a,b]$ is

$a = x_0 \leq x_1 \leq \ldots \leq x_n = b$.

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

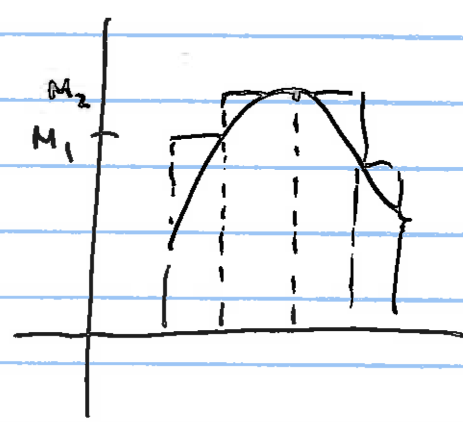
$A = \alpha(b) - \alpha(a)$

f bounded: $\exists m, M$ s.t. $m \leq f(x) \leq M$ $\forall x \in [a,b]$.



Let $m_i = \inf\limits_{x_{i-1} \leq x \leq x_i} f(x)$

Let $M_i = \sup\limits_{x_{i-1} \leq x \leq x_i} f(x)$



$P: \quad x_0 = a \le x_1 \le x_2 \le \dots \le x_n = b$

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

Define
$$\begin{cases} U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \\ L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \end{cases}$$

(Special Case: $\alpha(x) = x$ is the Riemann Sums)

$$m \cdot A \le L(P,f,\alpha) \le U(P,f,\alpha) \le M \cdot A$$

$\forall x \quad m \le f(x) \le M$

why?

$$\sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} M \Delta\alpha_i = M \sum_{i=1}^{n}\left(\alpha(x_i) - \alpha(x_{i-1})\right)$$

$M_i \le M$
$\Delta\alpha_i \ge 0$

$$= M\left(\alpha(b) - \alpha(a)\right)$$
$$= MA.$$

$\underline{\int_a^b} f\,d\alpha = \sup_P L(P,f,\alpha)$ exists

$\overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha)$ exists

} LUB prop.

A function f is RS integrable wrt α if

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha.$$ Notation $f \in \mathcal{R}(\alpha)$

* The common value is denoted by $\int_a^b f\,d\alpha$, when $f \in \mathcal{R}(\alpha)$

Ex

1) $f(x): [0,1] \to [0,1]$ $\qquad f(x) = x^2$

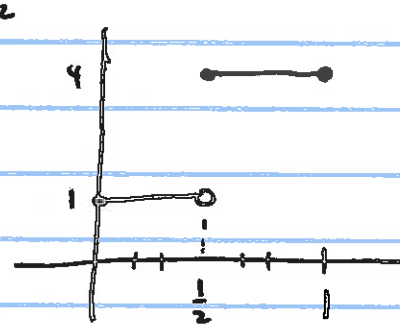
$\alpha(x): [0,1] \to \mathbb{R}$ $\qquad \alpha(x) = 2x+1$

$$\int_0^1 f\,d\alpha = \int_0^1 x^2 (2dx) = \frac{2}{3}$$

$$\Delta\alpha = 2\Delta x$$

2) $f(x) = x^2$

$\beta(x)$



$$\beta(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 4, & \frac{1}{2} \le x \le 1 \end{cases}$$

$\Delta\beta_i = 0$ if $\quad x_{i-1} \le x_i < \frac{1}{2}$

if $\quad \frac{1}{2} \le x_{i-1} \le x_i$

$\Delta\beta_i = 3$ if $\quad x_{i-1} < \frac{1}{2} \le x_i$

$$\int_0^1 f\,d\beta = 3f\left(\tfrac{1}{2}\right) = \frac{3}{4}.$$

Defn ⓒ A partition P^* is called a refinement P if

(i) both are partitions of $[a,b]$.

(ii) $P \subseteq P^*$

ⓑ Given two partitions P_1 and P_2 of $[a,b]$.

$P^* = P_1 \cup P_2$ is a common refinement of both P_1 & P_2.

Prop if $P \subseteq P^*$ (of partitions of $[a,b]$) then

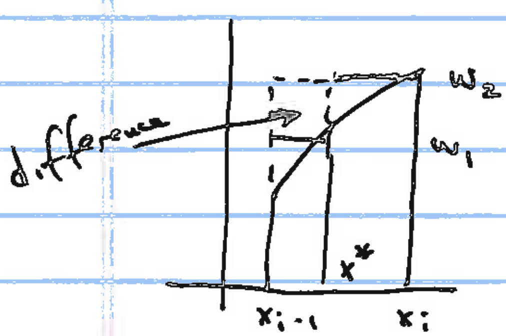
$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

Proof: **Case 1** $P^* - P = \{x^*\}$ a single pt.

Say $x^* \in (x_{i-1}, x_i)$ for some i.

$P: \; a = x_0 \leq x_1 \leq \ldots \leq x_i \leq x_{i+1} \leq \ldots \leq x_n = b$

$P^*: \; a = x_0 \leq x_1 \leq \ldots \leq x_i \leq x^* \leq x_{i+1} \leq \ldots \leq x_n = b$



$$W_1 = \sup_{x_{i-1} \leq x \leq x^*} f$$

$$W_2 = \sup_{x^* \leq x \leq x_i} f$$

$\max(W_1, W_2) = M_i$

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f$$

$U(P,f,\alpha) - U(P^*,f,\alpha)$

$$= M_i(\alpha(x_i) - \alpha(x_{i-1})) - \left[W_1(\alpha(x^*) - \alpha(x_{i-1})) + W_2(\alpha(x_i) - \alpha(x^*))\right]$$

(add & subtract $\pm M_i(\alpha(x^*))$)

$$= (M_i - W_1)(\alpha(x^*) - \alpha(x_{i-1})) + (M_i - W_2)(\alpha(x_i) - \alpha(x^*)]$$

$\geq 0 \qquad \geq 0 \qquad \geq 0 \qquad \geq 0$

≥ 0

$$U(P, f, \alpha) - U(P^*, f, \alpha) \geq 0$$

General Case $\quad P \subseteq P^*$.

Since both P & P^* are finite sets, $\exists\, P_1, P_2, \ldots P_\ell$ s.t.

(1) $\quad P = P_1 \subseteq P_2 \subseteq \ldots \subseteq P_\ell = P^*$

(2) $\forall i.\quad P_{i+1} - P_i$ consists one pt.

$$U(P, f, \alpha) \geq U(P_1, f, \alpha) \geq U(P_2, f, \alpha) \geq \ldots \geq U(P_\ell, f, \alpha) = U(P^*, f, \alpha)$$

by repeated application of Case 1.

Proof for $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ is very similar & it is in the book.