MVT for

VECTOR VALUED (single Variable) functions?

Let $f: [a,b] \subseteq \mathbb{R}^1 \to \mathbb{R}^k$

$f(t) = (f_1(t), f_2(t), \ldots, f_k(t))$

$i = 1, \ldots k \qquad f_i : [a,b] \to \mathbb{R}^1$

f is diff'ble $\Leftrightarrow$ each f_i is diff'ble.

Ex 1 MVT is false for $k > 1$.

$f(t) = (\cos t, \sin t) : [0, 2\pi] \to \mathbb{R}^2$

$f(0) = (1,0) = f(2\pi)$

$f'(t) = (-\sin t, \cos t)$

$\exists$? any $c \in (a,b)$: $f(b) - f(a) = f'(c) \cdot (b-a)$??

$(0,0) = f(2\pi) - f(0) \neq \underbrace{(-\sin c, \cos c)}_{\neq (0,0)} \cdot (2\pi) \neq (0,0)$

How did this happen?

MVT $f_1 \qquad f_1(b) - f_1(a) = f_1'(c_1) \cdot (b-a)$

MVT $f_2 \qquad f_2(b) - f_2(a) = f_2'(c_2)\ (b-a)$

No guarantee $c_1 = c_2$.

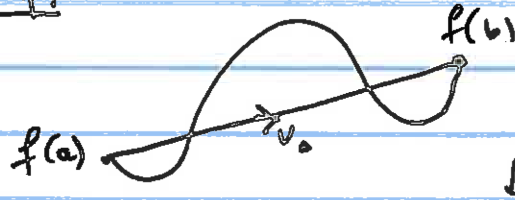
<u>Thm:</u> Let $\vec{f} : [a,b] \to \mathbb{R}^k$ be continuous

f be diff'ble on (a,b)

Then $\exists\, c \in (a,b)$

$$|f(b) - f(a)| \leq (b-a)\,|f'(c)|$$

<u>Proof:</u>



Let $\vec{v}_0 = f(b) - f(a)$

Let $\varphi(t) = \vec{v}_0 \cdot \vec{f}(t) : [a,b] \to \mathbb{R}^1$

$\varphi'(t) = \vec{v}_0 \cdot f'(t)$

Can Apply MVT (for $n=1$) to φ.

$\exists\, c \in (a,b)$ $\quad \varphi(b) - \varphi(a) = \varphi'(c)(b-a)$

① $$|\varphi(b) - \varphi(a)| = |\vec{v}_0 \cdot f(a) - \vec{v}_0 \cdot f(b)|$$
$$= |v_0 \cdot (\underbrace{f(a) - f(b)}_{v_0})| = |v_0|^2$$

$$|\varphi'(c)(b-a)| = |(v_0 \cdot f'(c))(b-a)|$$

$$= |b-a|\,|v_0 \cdot f'(c)|$$

② $$\leq |b-a|\,|v_0|\,|f'(c)|$$

③ $$|\varphi(b) - \varphi(a)| = |\varphi'(c)(b-a)|$$

$$|v_0|^2 \leq |b-a|\,|v_0|\,|f'(c)|$$

$$|f(b) - f(a)| = |v_0| \leq |b-a|\,|f'(c)|$$

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Taylor Polynomials

Def^n: Let $f : [a,b] \longrightarrow \mathbb{R}^1$ be given, $\alpha \in [a,b]$.

Let $m \in \mathbb{N}$ s.t.

$f', f'', f''', \ldots, f^{(m)}$ are defined on $[a,b]$

$$P_{\alpha,m} = \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$ is called

the $m^{\underline{th}}$ degree Taylor polynomial of f at α.

Taylor's Thm: Let $f : [a,b] \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$ s.t.

$f^{(n-1)}$ is continuous on $[a,b]$

$f^{(n)}$ exists on (a,b)

Let $\alpha \in (a,b)$, $\beta \in [a,b]$. Then

$$f(\beta) = P_{n-1,\alpha}(\beta) + \frac{f^{(n)}(c)}{n!}(\beta-\alpha)^n$$ for some c between α & β.

Obs (1) $f^{(n-1)}$ continuous $\Longrightarrow$ $f', f'', f''', \ldots f^{(n-2)}$ all exist & continuous

(2) For $n=1$, Taylor's Thm is MVT:

$P_{1,\alpha}(\alpha) = f(\alpha)$ $f(\beta) = f(\alpha) + f'(c)(\beta-\alpha)$ MVT.

4

WTS: $f(\beta) = P_{n-1,\alpha}(\beta) + \dfrac{f^{(n)}(c)}{n!}(\beta-\alpha)^n$; $\left\{ \begin{array}{l} \text{for some} \\ c \text{ between} \\ \alpha \,\&\, \beta. \end{array} \right.$

<u>Proof:</u> for $n \geq 2$

Choose M s.t.

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n \qquad \text{For } (\beta \neq \alpha)$$

$$g(t) = f(t) - P(t) - M(t-\alpha)^n$$

$$g(\alpha) = f(\alpha) - P(\alpha) - M(\alpha-\alpha)^n = f(\alpha) - f(\alpha) = 0.$$

$$g(\beta) = 0$$

By MVT for g $\exists\, \beta_1 \in$ between $\alpha \,\&\, \beta$ $\quad g'(\beta_1) = 0$

$$g'(t) = f'(t) - P'(t) - Mn(t-\alpha)^{n-1}$$

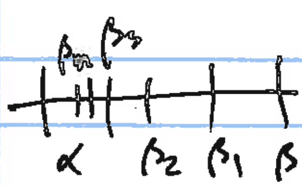
$$= f'(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} k(t-\alpha)^{k-1} - Mn(t-\alpha)^{n-1}$$

$$= f'(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-1)!}(t-\alpha)^{k-1} - Mn(t-\alpha)^{n-1}$$

$g'(\beta_1) = 0 \qquad g'(\alpha) = f'(\alpha) - f'(\alpha) = 0$

By MVT $\exists\, \beta_2$ between $\alpha \,\&\, \beta_1$

s.t. $\quad g''(\beta_2) = 0$



$\exists\, \beta_3$ between $\alpha \,\&\, \beta_2$ s.t.

$g'''(\beta_3) = 0$

$\Rightarrow \exists\, \beta_n$ s.t. $\quad g^{(n)}(\beta_n) = 0$ between $\alpha \,\&\, \beta_{n-1}$

$g^{(n)} = f^{(n)}(t) - n!\,M \qquad$ Take $c = \beta_n$,

$g^{(n)}(c) = 0 \Rightarrow M = \dfrac{f^{(n)}(c)}{} (\beta-\alpha)^n.$ #

Ex. Why do we care about $M = \frac{f^{(n)}(c)}{n!}(\beta-\alpha)^n$?

$$\underbrace{|f(\beta) - P(\beta)|}_{\text{error}} \le \frac{f^{(n)}(c)}{n!}(\beta-\alpha)^n$$

Error estimate for the approximation of $f(\beta)$ by $P_{n-1}(\beta)$.

If $|f^{(n)}(x)| \le C$ for some C, $\forall x \in [\alpha, \beta]$ then

$$|\text{error}| \le \frac{C}{n!}(\beta-\alpha)^n.$$

Easy example $f(x) = e^x$

$$P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \qquad n-1=3, \; n=4.$$

$$e^{0.01} \approx 1 + 0.01 + \frac{(0.01)^2}{2} + \frac{(0.01)^3}{6}$$

$$= 1.0100501\bar{6}$$

$$|\text{Error}| \le \frac{3}{4!}(0.01)^4 \quad \text{since } |e^x| \le e \le 3 \text{ for } 0 \le x \le 1.$$

$$= 0.00000000125$$

$$e^{0.01} \approx 0.0100501666 6\ldots$$
$$\pm 0.00000000125\ldots$$