April 4, 2018

1

$f : (a,b) \to \mathbb{R}$, diff'ble.

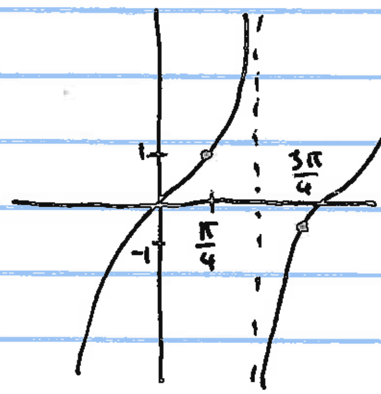
Ex 1 $\forall x<y \quad f(x)<f(y) \quad \not\Leftrightarrow \quad f'(x)>0 \quad \forall x$

$f(x) = x^3$, $\quad x<y \Rightarrow x^3<y^3$

but $\quad f'(0)=0$.

2

$$f(x) = \tan x : \underbrace{\mathbb{R} - \left\{ \tfrac{\pi}{2} + n\pi \;\middle|\; n \in \mathbb{Z} \right\}}_{\text{not connected domain.}} \longrightarrow \mathbb{R}.$$



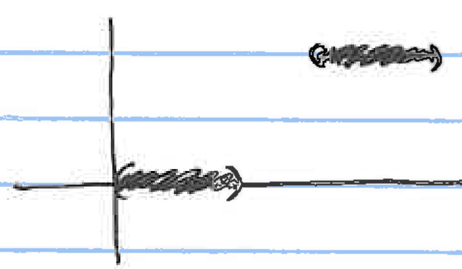
$f'(x) = \sec^2 x > 0$

$\frac{\pi}{4} < \frac{3\pi}{4}$ but

$1 = \tan\frac{\pi}{4} \not< \tan\frac{3\pi}{4} = -1$

3

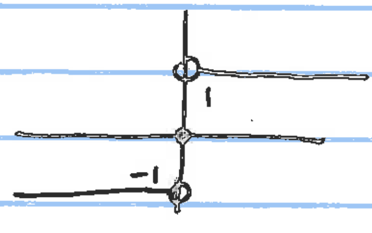
$$g(x) = \begin{cases} 0 & 0<x<1 \\ 2 & 2<x<3 \end{cases}$$

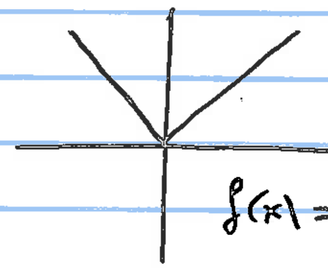


$g'(x) = 0$ on $(0,1)\cup(2,3)$

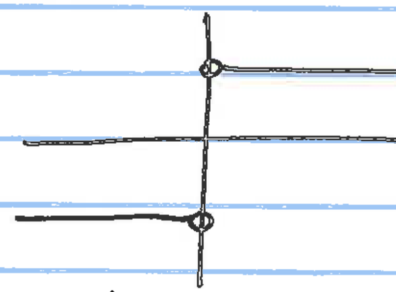
$g(x) \neq$ constant on $(0,1)\cup(2,3)$

Not every function is a derivative





$f'(x) \neq g(x)$



g is not the derivative of f, but is g the derivative of another function?

(Darboux) Thm: Let $f : [a, b] \longrightarrow \mathbb{R}$ be diff'ble (on all of $[a,b]$). If $f'(a) < \lambda < f'(b)$ then $\exists\, c \in (a,b)$ s.t. $f'(c) = \lambda$.

Caution: This is not obtained by applying Intermediate Value Thm to f', since we do not know that f' is continuous.

$$f'(a) < \lambda < f'(b)$$

Proof: Put $g(t) = f(t) - \lambda t$

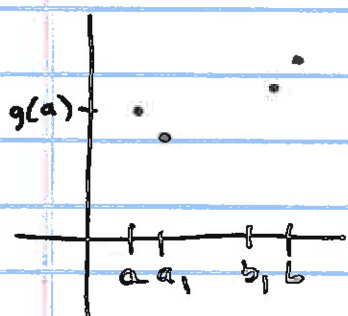
$g'(t) = f'(t) - \lambda.$

$g'(a) = f'(a) - \lambda < 0.$

$$\frac{g(t) - g(a)}{t - a} \to g'(a) < 0$$

By Lemma *

$\exists \delta > 0 \quad a < t < a + \delta \quad \dfrac{g(t) - g(a)}{t - a} < 0$



$g(t) - g(a) < 0$

$g(t) < g(a)$

say $a_1 \in (a, a+\delta)$

$g(a_1) < g(a)$

$g'(b) = f'(b) - \lambda. > 0$

$$\frac{g(t) - g(b)}{t - b} \to g'(b) > 0$$

By Lemma *

$\exists \delta' > 0 \quad b - \delta' < t < b \quad \dfrac{g(t) - g(b)}{t - b} > 0$

$g(t) - g(b) < 0$

$g(t) < g(b)$

$\exists b_1 \in (b - \delta', b) \quad g(b') < g(b)$

Minimum of g must be in (a, b), say at $c \in (a, b)$

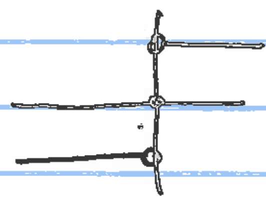
g diff'ble on (a, b)

$\Rightarrow \quad g'(c) = 0$

$0 = g'(c) = f'(c) - \lambda \implies f'(c) = \lambda.$

Corollary: If f is diff'ble on <u>all</u> of $[a,b]$, then f' cannot have any discontinuity of the first kind.
Consequently all discontinuities of f' must be of the second kind.

Ex



is not the derivative of any function.

L'Hospital's Rule

Let $f, g : (a,b) \to \mathbb{R}$ be diff'ble, $g'(x) \neq 0$ on (a,b)
where $-\infty \leq a < b \leq \infty$

If (i) $\dfrac{f'(x)}{g'(x)} \longrightarrow A$ as $x \to a$

and

(ii) either (a) $\left.\begin{array}{l} f(x) \to 0 \\ g(x) \to 0 \end{array}\right\}$ as $x \to a$

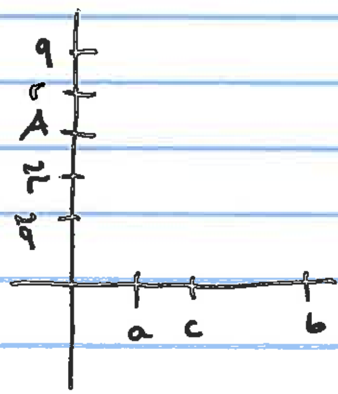
or

(b) $g(x) \to \infty$ as $x \to a$

Then $\dfrac{f(x)}{g(x)} \longrightarrow A$, as $x \to a$.

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = A.$$

Proof of the case (i) & (ii a) $\begin{cases} -\infty < a < b < \infty \\ -\infty < A < \infty \end{cases}$



$\forall r, q \quad A < r < q \quad \exists c \in (a,b)$

$\dfrac{f'(x)}{g'(x)} < r$ for all $a < x < c$

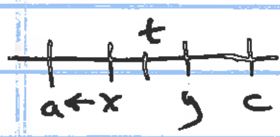
(Def of limit)

$\forall x, y \quad a < x < y < c$

$\exists t \in (x,y)$

By GMVT $\quad \dfrac{f(x) - f(y)}{g(x) - g(y)} = \dfrac{f'(t)}{g'(t)} < r$

Let $x \to a$ $\quad \begin{pmatrix} f(x) \to 0 \\ g(x) \to 0 \end{pmatrix}$



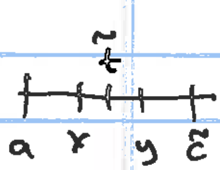
$\dfrac{f(x) - f(y)}{g(x) - g(y)} \longrightarrow \dfrac{f(y)}{g(y)} \leq r < q.$

$\forall y \quad a < y < c.$

$\forall \tilde r, \tilde q \quad \tilde q < \tilde r < A \quad \exists \tilde c \in (a,b)$

$\tilde r < \dfrac{f'(x)}{g'(x)} \quad \forall x \quad a < x < \tilde c$

$\forall x, y \quad a < x < y < \tilde c \quad \exists \tilde t \quad x < \tilde t < y$



(GMVT) $\quad \dfrac{f(x) - f(y)}{g(x) - g(y)} = \dfrac{f'(\tilde t)}{g'(\tilde t)} > \tilde r$

again $x \to a \quad \dfrac{f(y)}{g(y)} \geq \tilde r > \tilde q$

Hence $\forall\ \tilde{q} < A < q \ \exists\ c, \tilde{c}$ s.t

$$\tilde{q} < \frac{f(y)}{g(y)} < q \quad \forall\, y \in (a, \min(c, \tilde{c}))$$

$$\lim_{y \to a} \frac{f(y)}{g(y)} = A.$$

(i) & (iib) Case: $g(x) \to \infty$ ($f(x) \to \infty$ <u>not</u> needed)

Same as in case (i)

$\forall\, r, q \quad A < r < q \ \exists\ c \in (a,b), \ \frac{f'(x)}{g'(x)} < r$

$\forall\, x, y \quad a < x < y < c \ \exists\ t \in (x,y)$

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r$$

Before $g(x) \to \infty$, we need to do some algebra

$$\frac{g(x) - g(y)}{g(x)} \frac{f(x) - f(y)}{g(x) - g(y)} < \frac{g(x) - g(y)}{g(x)} \cdot r$$

$$\frac{f(x) - f(y)}{g(x)} < \left(1 - \frac{g(y)}{g(x)}\right) r$$

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}$$

Now let $x \to a$. $g(y), f(y)$ are fixed

$g(x)$ gets large, and

$$\left| -r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \right| < q - r$$

$\Rightarrow \ \frac{f(x)}{g(x)} < q$. for $x \in (a, c_1)$ for some $c_1 \leq c$.

Similarly $\tilde{q} < \frac{f(x)}{g(x)}$ for $\forall\, x \in (a, \tilde{c}_1)$. Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = A$.