1/22/18

1

Chap I Continue

Extended Real #'s : $\mathbb{R} \cup \{+\infty, -\infty\}$

$\forall x \in \mathbb{R} \quad -\infty < x < \infty$

∞ becomes an upper bd for all subsets of $\mathbb{R}$.

If E is not bdd above : $\sup E = +\infty$.

$\sup \emptyset = -\infty$

Complex #'s.

On $\mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$

Define $\begin{cases} (a,b) + (c,d) = (a+c, b+d) \\ (a,b)\cdot(c,d) = (ac-bd, bc+ad) \end{cases}$

$\mathbb{R}^2$ becomes a field.

$(1,0) = 1$

$(0,1) = i$

$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$

HW to read p 12 - 15.

Euclidean Spaces:

$$\mathbb{R}^k = \{(x_1, x_2, \dots x_k) \mid x_i \in \mathbb{R} \ \forall i = 1, \dots k\}$$

$\mathbb{R}^k$ is a $\mathbb{R}$-vector space: $a \cdot (x_1, \dots x_k) = (ax_1, \dots, ax_k)$

$$(x_1, x_2, \dots x_k) + (y_1, y_2 \dots y_k) = (x_1 + y_1, \dots, x_k + y_k)$$

<u>Defn</u> $\vec{x} \cdot \vec{y} = (x_1, x_2 \dots x_k) \cdot (y_1, \dots y_k) = \sum_{i=1}^{k} x_i y_i$

$$\|\vec{x}\| = |\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}$$

<u>Thm:</u> $|\vec{x}| \geq 0$

$|\vec{x}| = 0 \iff \vec{x} = \vec{0}$

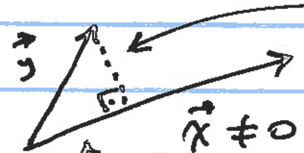
$|a\vec{x}| = |a| |\vec{x}|$

<u>will prove:</u>

① $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| |\vec{y}|$ Cauchy-Schwarz

② $|x + y| \leq |x| + |y|$ Δ-inequality

<u>Proof ①</u>



$$\underbrace{\frac{\vec{y} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \vec{x}}_{\text{orthogonal projection}}$$

$$\underbrace{y - \frac{y \cdot x}{x \cdot x} x}_{\text{normal component to } \vec{x}}$$

Will not use "$\rightarrow$" to avoid cluttering

WLOG: $x \neq 0$

$$0 \leq \left| y - \frac{y\cdot x}{x\cdot x} x \right|^2 = \left(y - \frac{y\cdot x}{x\cdot x} x\right)\cdot\left(y - \frac{y\cdot x}{x\cdot x} x\right)$$

$$= y\cdot y - 2\, y\cdot \frac{y\cdot x}{x\cdot x} x + \left(\frac{y\cdot x}{x\cdot x}\right)^2 \cdot x\cdot x$$

$$= |y|^2 - 2(y\cdot x)^2 / x\cdot x + (y\cdot x)^2 / x\cdot x$$

$$= |y|^2 - (y\cdot x)^2 / |x|^2$$

$$\Longrightarrow \quad 0 \leq |y|^2 - \frac{(y\cdot x)^2}{|x|^2}$$

$$\frac{(y\cdot x)^2}{|x|^2} \leq |y|^2$$

$$(y\cdot x)^2 \leq |x|^2 |y|^2$$

$$|y\cdot x| \leq |x||y|. \quad \#$$

Remark: This proof works in any inner product space, even if infinite dimensional

ex $\quad f, g : [a,b] \xrightarrow{C^0} \mathbb{R}$

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx \qquad \langle f, f\rangle = \int_a^b f^2(x)\, dx = \|f\|_{L^2}^2$$

$$\left| \int_a^b f(x)\, g(x)\, dx \right| \leq \left(\int_a^b f^2\, dx\right)^{1/2} \left(\int_a^b g^2\, dx\right)^{1/2}$$

Its proof is same/similar to the proof above

Proof of Δ-inequality

(2) $\|x+y\| \leq \|x\| + \|y\|$

Proof: $|x+y|^2 = (x+y)\cdot(x+y)$

$= x\cdot x + 2x\cdot y + y\cdot y$

$= |x|^2 + 2x\cdot y + |y|^2$

use Cauchy-Schwarz

$|x\cdot y| \leq |x||y|$

$\leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$

$|x+y|^2 \leq (|x| + |y|)^2$

$|x+y| \leq (|x| + |y|)$. #

End of Ch. I.

Chap II:

(2.1) Def^n Let A and B be sets.

A function f from A to B is an assignment of a value $f(x) \in B$ for each $x \in A$.

A: Domain of f
B: Codomain of f
$\{f(x) \mid x \in A\}$ Range of f

Notation $f: A \to B$.

Def^n Let $f: A \to B$

For $E \subseteq A$, $\{f(x) \mid x \in E\} = f(E)$ image of E under f
For $E \subseteq B$, $\{x \in A \mid f(x) \in E\} = f^{-1}(E)$
pre-image or inverse-image of E under f.

f is called onto if $f(A) = B$ (surjective)
f is called one-to-one if (injective)

equivalent
$\forall x_1, x_2 \in A\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$ OR
$\forall x_1, x_2 \in A\ (x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$ OR
$\forall y \in B$ $f^{-1}(y)$ has at most one element.

Def^n $f: A \to B$ is called a bijection if f is both 1-1 and onto. If that's the case,

then we can say also
- f is called a one-to-one correspondence between A & B.
- $A \sim B$, A & B are equivalent
- A & B have the same cardinality

Notation $J_n = \{1, 2, 3, \dots n\}$

$J = \mathbb{N} = \{1, 2, 3, \dots n, \dots\}$

Defn A set A is called....

- finite if $A \sim J_n$ for some $n \in \mathbb{N}$.
- infinite if A is not finite
- countable if $A \sim \mathbb{N}$.
- uncountable if A is neither finite nor countable
- at most countable if A countable or finite.

we will use the book's notation.

CAUTION This is different from the common usage today

$A \sim \mathbb{N}$	countably infinite	Many people use these today.
(finite or $A \sim \mathbb{N}$) $\longleftrightarrow$	countable	
uncountable $\longleftrightarrow$	not countable	

Ex $\mathbb{N}$ infinite, countable.

$\mathbb{Z}$ " "

$\exists f : \mathbb{N} \to \mathbb{Z}$ bijection

$$\begin{array}{cccccccc} & 0, & 1, & -1, & 2, & -2, & 3, & -3 \\ f & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

$$f(n) = \begin{cases} n/2 \\ -\left(\frac{n-1}{2}\right) \end{cases}$$

$\mathbb{Q}$ is infinite, countable

$\mathbb{R}$ is uncountable