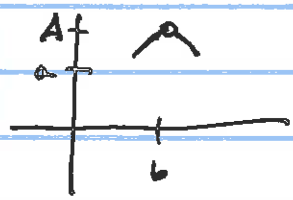
* <u>Lemma:</u> Let $f: I \to \mathbb{R}$, $b \in I$, $\lim_{x \to b} f(x)$ exists.

(i) If $\exists\, \delta > 0$ s.t. $f(x) \geq a$, for $x \in (b-\delta, b+\delta)$, $x \neq b$ $\Rightarrow \lim_{x \to a} f(x) \geq a$

(ii) If $\lim_{x \to b} f(x) > a \Rightarrow \exists\, \delta > 0$ $f(x) > a$ $\forall x \in (b-\delta, b+\delta)$, $x \neq b$

<u>Proof (ii)</u> Let $A = \lim_{x \to b} f(x) > a$



Let $\varepsilon = A - a$

$\exists\, \delta > 0$ s.t.

$0 < |x - b| < \delta \Rightarrow |f(x) - A| < \varepsilon = A - a.$

$-\varepsilon < f(x) - A < \varepsilon = A - a$

$a - A < f(x) - A < A - a$

$a < f(x).$

(ii) $\Rightarrow$ (i) By first proving (ii) for $< a$;
Assuming $\lim f(x) < a$, then giving a proof by obtaining a contradiction.

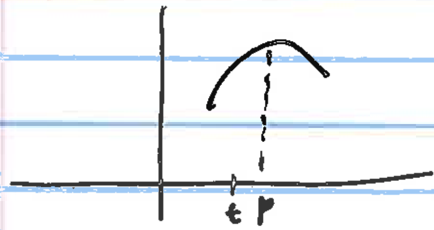
<u>Defn</u> $f: (X, d_X) \to \mathbb{R}$. f is said to have a local max at $p \in X$ if $\exists\, \delta > 0$ s.t. $\forall x \in N_\delta(p)$, $f(x) \leq f(p)$.

<u>Thm:</u> Let $f: [a,b] \to \mathbb{R}$. If f has a local max at p where $p \in (a,b)$ and if f is diff'ble at p then $f'(p) = 0$.

<u>Proof:</u> Choose $\delta > 0$ as in defn of local max
also $(p-\delta,\ p+\delta) \subseteq (a,b)$

$p-\delta < t < p \implies \dfrac{f(t)-f(p)}{t-p} \geq 0$ $\left(\text{since } f(p) \geq f(t) \text{ and } p > t\right)$

by Lemma # (i)



$f'(p) = \lim\limits_{t \to p} \dfrac{f(t)-f(p)}{t-p}$ $\left.\begin{matrix} \geq 0 \\ \leq 0 \end{matrix}\right\}$ has to be 0

by Lemma # (i)

$p < t < p+\delta \implies \dfrac{f(t)-f(p)}{t-p} \leq 0$ $\left(\text{since } f(p) \geq f(t) \text{ and } p < t\right)$

$((a \text{ and } b) \Rightarrow c) \equiv (a \Rightarrow (c \text{ or not } b))$

(p is interior pt & p is a local max & $f'(p)$ exist $\implies f'(p) = 0$)

equivalent (p is a local max $\implies$ $\left\{\begin{matrix} f'(p) = 0 \text{ OR } f'(p) \text{ DNE OR} \\ p \text{ is a boundary point.} \end{matrix}\right.$

<u>Mean Value Thms:</u>

(3770) <u>MVT</u>: Let $f : [a,b] \to \mathbb{R}$ be continuous, $a < b$
and f be diffble on (a,b).
then $\exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(a)-f(b)}{a-b}.$$

Take $g(x) = x$ $\Uparrow$

<u>GMVT</u>: Let $f, g : [a,b] \to \mathbb{R}$ be continuous, $a < b$
(Generalized) f, g be both diffble on (a,b),
then $\exists\, c \in (a,b)$ s.t.

$$(f(b)-f(a))\,g'(c) = (g(b)-g(a)) \cdot f'(c).$$

Proof of (GMVT)

Let $h(t) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$

h continuous on $[a,b]$

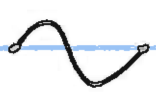
h diff'ble on (a,b)

$$h(a) = (f(b) - f(a))g(a) - (g(b) - g(a))f(a)$$
$$= f(b)g(a) - g(b)f(a)$$

$$h(b) = (f(b) - f(a))g(b) - (g(b) - g(a))f(b)$$
$$= -f(a)g(b) + g(a)f(b)$$

$h(a) = h(b)$.

Case 1 h is constant on $(a,b) \Longrightarrow h'(x) \equiv 0 \ \forall x \in (a,b)$



Case 2 $h(t) > h(a)$ for some $t \in (a,b)$, then $\exists c$ where h has a maximum at $c \in (a,b)$: $h'(c) = 0$

Case 3 $h(t) < h(a)$ for some $t \in (a,b)$, then $\exists c$ where h has a minimum at $c \in (a,b)$: $h'(c) = 0$

In all cases $\exists c \in (a,b)$ s.t. $h'(c) = 0$.

$$h'(t) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$$

$$0 = h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c). \quad \#$$

<u>Corollary</u> Let $f:(a,b)\to\mathbb{R}$ be diff'ble

(a) $f'(x)\geq 0\ \forall x\in(a,b) \iff f(x)\geq f(y)\ \left\{\begin{array}{l}\forall x,y\in(a,b),\\ x\geq y.\end{array}\right.$

(b) $f'(x)\leq 0\ \forall x\in(a,b) \iff f(x)\leq f(y)\ \left\{\begin{array}{l}\forall x,y\in(a,b),\\ x\geq y.\end{array}\right.$

(c) $f'(x)\equiv 0\ \forall x\in(a,b) \iff f\equiv \text{constant}$

(a) $\Rightarrow$: $\forall x\neq y$ $\dfrac{f(x)-f(y)}{x-y}=f'(c)$ for some c between x & y

by MVT

$f'(c)\geq 0$

$\dfrac{f(x)-f(y)}{x-y}\geq 0 \xrightarrow{x>y} f(x)-f(y)\geq 0$
$f(x)\geq f(y)$
when $x>y$

$\Leftarrow$: $f(x)\geq f(y)$
$x\geq y$

when $x\neq y$: $\dfrac{f(x)-f(y)}{x-y}\geq 0$

$f'(y)=\lim\limits_{x\to y}\dfrac{f(x)-f(y)}{x-y}\geq 0$ By Lemma *

(b) & (c) are similar.