March 30, 2018

1

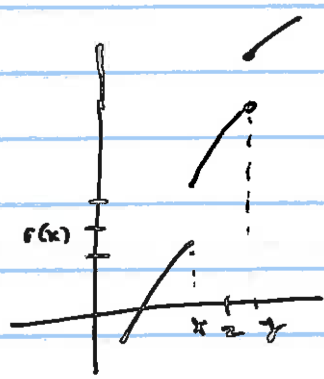
End of Chapter IV

Corollary 2 Monotonic functions have at most countable discontinuities.

Proof: Assume $f \uparrow$ $\quad f: I \longrightarrow \mathbb{R}$, I interval.

$$D = \{x \mid f \text{ is discontinuous at } x\}$$

$$\forall x \in \quad f(x-) < f(x+)$$



Let $r(x) \in \mathbb{Q}$ s.t.

$$f(x-) < r(x) < f(x+)$$

If $x, y \in D$, $x \neq y$ then $r(x) \neq r(y)$, since if $x < y$ $\exists z$ $x < z < y$, and

$$r(x) < f(x+) \leq f(z) \leq f(y-) < r(y)$$

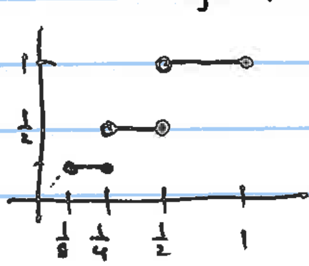
$r: D \to \mathbb{Q}$. 1-1 restrict the codomain.

$\tilde{r}: D \to r(D) \subseteq \mathbb{Q}$ $\quad \tilde{r}$ is a bijection

$\mathbb{Q}$ countable $\Longrightarrow$ $r(D)$, a subset of a countable set is at most countable.

$\tilde{r}$ bijection $\Longrightarrow$ D is at most countable.

Ex



$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^n & \text{if } \left(\frac{1}{2}\right)^{n+1} < x \leq \left(\frac{1}{2}\right)^n \\ 0 & \text{if } x = 0 \end{cases}$$

Bounded domain & range. ∞ly many disc.

Chap V DIFFERENTIATION

Defn Let $f:[a,b] \to \mathbb{R}$. For a given $x \in [a,b]$ fixed define

$$\phi(t) = \frac{f(t)-f(x)}{t-x} \quad \text{for} \quad a<t<b, \; t \neq x.$$

define $f'(x) = \lim_{t \to x} \phi(t)$, if it exists.

f' is called the derivative, when defined
if $f'(x)$ is defined then f is called diff'ble at x.
f is called diff'ble if f is diff'ble at all $x \in [a,b]$.
In general (domain f') $\subseteq [a,b]$.

(5.2) **Thm:** Let $f: [a,b] \to \mathbb{R}$
f is diff'ble at $x \implies f$ is continuous at x.

Proof: $f(t) - f(x) = \frac{f(t)-f(x)}{t-x} \cdot (t-x) \xrightarrow[\text{as } t \to x]{} f'(x) \cdot 0 = 0$
for $t \neq x$.

$$\left.\begin{array}{l} f(t) - f(x) \to 0 \\ f(t) \to f(x) \end{array}\right\} \text{ as } t \to x.$$

Thm 5.3, 5.4 HW to read.

Chain Rule

Thm 5.5 Let $f : [a,b] \to \mathbb{R}$ continuous.

$g : \underset{\text{interval}}{I} \to \mathbb{R}$, $I \supseteq f([a,b])$

f is diff'ble at x

g is diff'ble at $y = f(x)$

Then $h(t) = g(f(t))$ is diff'ble at x & $h'(x) = g'(f(x)) \cdot f'(x)$

$$[a,b] \xrightarrow{f} I \longrightarrow \mathbb{R}$$

Proof:

$$\frac{f(t) - f(x)}{t - x} - f'(x) = u(t) \qquad x \text{ is fixed}$$

rewrite

$$f(t) - f(x) = (t - x)(f'(x) + u(t))$$

$$g(s) - g(y) = (s - y)(g'(y) + v(s))$$

$$u(t) \to 0 \quad \text{as } t \to x$$

$$v(s) \to 0 \quad \text{as } s \to y = f(x)$$

set $s = f(t)$

$$h(t) - h(x) = g(\underbrace{f(t)}_{s}) - g(\underbrace{f(x)}_{y}) = (f(t) - f(x))(g'(y) + v(s))$$

$$= (t - x)(f'(x) + u(t)) \cdot (g'(y) + v(s))$$

$$\frac{h(t) - h(x)}{t - x} = (f'(x) + u(t))(g'(y) + v(s)) \longrightarrow f'(x) \cdot g'(y)$$

as $t \to x$: $f'(x) + u(t) \to f'(x)$, $g'(y) + v(s) \to g'(y)$

$$f'(x) \cdot g'(y) = f'(x) \cdot g'(f(x))$$

Since $s = f(t)$

$t \to x \Rightarrow \underbrace{f(t) \to f(x)}_{f \text{ continuous}}$

Ex
$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t\to 0} \frac{t^2 \sin\frac{1}{t} - 0}{t - 0}$$

$$= \lim_{t\to 0} t \sin\frac{1}{t} = 0$$

$\forall \varepsilon > 0 \ \exists \delta = \varepsilon \ \forall t, \ 0 < |t - 0| < \delta = \varepsilon$

$\left|t \sin\frac{1}{t} - 0\right| = |t|\left|\sin\frac{1}{t}\right| \leq |t| < \delta = \varepsilon.$

$x \neq 0$ $\quad f'(x) = 2x \sin\frac{1}{x} + x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$

$$f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}.$$

$\lim_{x\to 0} f'(x) = \underline{\text{DNE}}$ $\quad$ since $\underbrace{2x \sin\frac{1}{x} \to 0}_{\text{similarly to above}}$ but $\lim_{x\to 0} \cos\frac{1}{x}$ DNE! $\underline{\text{Why?}}$

$\Uparrow$

$t_n = \frac{1}{2n\pi} \to 0 \qquad \cos(1/t_n) = \cos 2\pi n \to 1$

$s_n = \frac{1}{(2n+1)\pi} \to 0 \qquad \cos(1/s_n) = \cos(2n+1)\pi \to -1$