March 28, 2018

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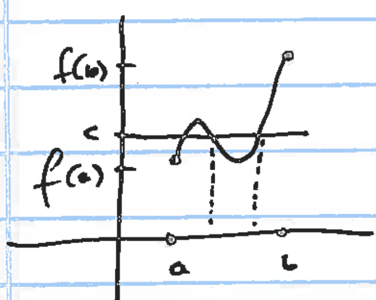
Recall Theorem 2.47

$\left.\begin{array}{l} E \subseteq \mathbb{R} \\ E \text{ connected} \end{array}\right\} \iff \forall x, y \in E,\ x < z < y \Rightarrow z \in E.$

Thm: Intermediate Value Thm

Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous.

If $f(a) < c < f(b)$, then $\exists\, x \in (a,b)$ s.t. $f(x) = c$



Proof: $[a,b] \subseteq \mathbb{R}$ is connected.

f continuous

$J = f([a,b])$ is connected.

$f(a), f(b) \in J.$

If c s.t. $f(a) < c < f(b)$, then $c \in J = f([a,b])$

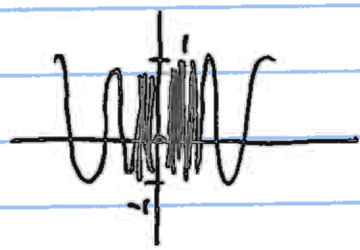
$\exists\, \underset{[a,b]}{\overset{\uparrow}{x}}$ s.t. $f(x) = c$

$x \neq a$ since $f(a) \neq c = f(x)$

$x \neq b$ since $f(b) \neq c = f(x)$

So $x \in (a,b)$.

Ex $f(x) = \begin{cases} \sin 1/x & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



$\lim_{x \to 0} \sin \frac{1}{x}$ DNE.

$f(0) = 0$

- f satisfies $\forall a, b \in \mathbb{R}$ if $f(a) < c < f(b)$ $\Rightarrow \exists x \in (a,b)$ s.t. $f(x) = c$

but f not continuous (Converse of IVT is false)

- $f([a,b]) = [c,d]$ but f is not continuous

$a, b > 0$ answer follows easily for f being continuous on $(0, \infty)$

$a, b < 0$ " " " " " " $(-\infty, 0)$

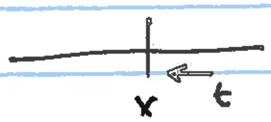
if $\left.\begin{array}{l} a < 0 < b \\ a = 0 < b \\ a < 0 = b \end{array}\right\}$, then $f([a,b]) = [-1,1]$

Hence $\underset{\forall a,b}{f([a,b]) = [c,d]} \not\Rightarrow f$ continuous.

DISCONTINUITIES

Defn $f:(a,b)\to\mathbb{R}$, For $x\in[a,b)$

$$f(x+)=\lim_{t\to x^+} f(t)=q$$



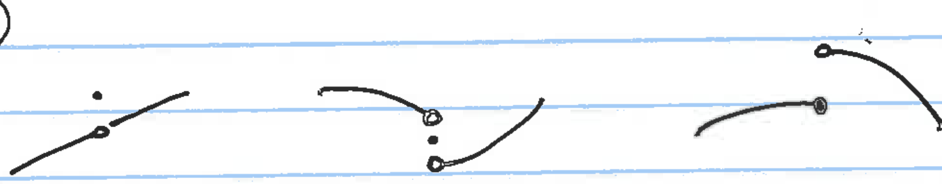
$$\iff \forall x_n\to x,\ x_n>x,\ f(x_n)\to q$$

$$\iff \forall \varepsilon>0\ \exists\delta>0$$

$$x<t<x+\delta \implies |f(t)-q|<\varepsilon$$

Defn Let $f:(a,b)\to\mathbb{R}$. Let $x\in(a,b)$ s.t. f is not continuous at x.
f is said to have a discontinuity of the first kind (simple discontinuity) if $f(x+)$, $f(x-)$ exist.
All others will be called discontinuities of 2nd kind.

Ex



Ex $f(x)=\begin{cases}0 & \text{if } x\in\mathbb{Q}\\ 1 & \text{if } x\notin\mathbb{Q}.\end{cases}$

- discontinuous on all of $\mathbb{R}$
- All " are of the 2nd kind

$g(x)=\begin{cases}x & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}.\end{cases}$

continuous only at 0
everywhere else discontinuity of 2nd kind

Defn A function $f:(a,b)\to\mathbb{R}$ is called monotonically increasing if $\forall\ a<x<y<b$, $f(x)\le f(y)$

: monotonically decreasing if $\forall a<x<y<b$, $f(x)\ge f(y)$

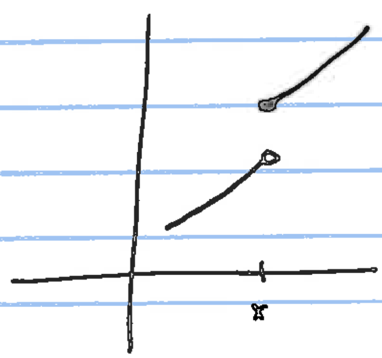
: monotone if it is either monotonically increasing or " decreasing.

Thm: Let f be monotonically $\nearrow$ on (a,b). Then

1) $$\sup_{a<t<x} f(t) = f(x-) \le f(x) \le f(x+) = \inf_{x<t<b} f(t)$$

2) If $a<x<y<b$, then

$$f(x+) \le f(y-)$$



Proof PTO

Proof: $f \uparrow$.

$\{f(t) \mid a<t<x\}$ is bounded above by $f(x)$

$\exists\, A = \sup\{f(t) \mid a<t<x\}$, $A \le f(x)$ (LUB prop.)

Want to show $A = f(x-)$

Let $\varepsilon>0$ be given, then $A-\varepsilon$ is not an upper bd for $\{f(t) \mid a<t<x\}$.

$\exists\, t \in (a,x)$ s.t. $A-\varepsilon < f(t) \le A$.

Set $\delta = x-t$. $A-\varepsilon < f(x-\delta) \le A$.

$\forall t \quad x-\delta < t < x, \quad A-\varepsilon < f(x-\delta) \underset{f\uparrow}{\le} f(t) \le A \le f(x)$

$\Rightarrow \forall t \quad x-\delta < t < x \quad A-\varepsilon < f(t) \le A$

$|f(t)-A| < \varepsilon$

We established that $\forall \varepsilon>0\ \exists\, \delta>0$ s.t.

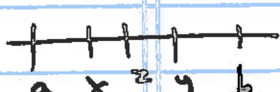
$\forall t \quad x-\delta<t<x \Rightarrow |f(t)-A|<\varepsilon$

by defn, same as: $f(x-) = A$.

$$f(x-) = \lim_{t\to x^-} f(t) = A = \sup_{a<t<x} f(t) \le f(x)$$

Similarly $$f(x+) = \lim_{t\to x^+} f(t) = B = \inf_{x<t<b} f(t) \ge f(x) \quad (1)$$ is done.

(2) $x<y \Rightarrow \exists\, z, \quad x<z<y$



$$f(x+) = \inf_{x<t<b} f(t) \le f(z) \le \sup_{a<t<y} f(t) = f(y-).$$

since $z \in (x,b)$ $\qquad$ $z \in (a,y)$

Corollary: Monotonic functions have discontinuities of the first kind only.