① Quiz #4 is on Chap 3.

② HW #11 (Chap 4) do not do part involving #13.

4.19 **Thm:** Let $f: (X, d_X) \longrightarrow (Y, d_Y)$ be continuous, and let X be compact.
Then f is uniformly continuous.

Proof: Let f be continuous.
Let $\varepsilon > 0$ be given.
For a given p,
f continuous at $p \Rightarrow \exists \delta_p > 0$ s.t.

$$(*) \quad \forall q \in X, \ d_X(q,p) < \delta_p \Longrightarrow d_Y(f(q), f(p)) < \frac{\varepsilon}{2}$$

Let $V_p = \{ q \in X \mid d_X(q,p) < \frac{\delta_p}{2} \}$; it is an open set.

$$X \subseteq \bigcup_{p \in X} V_p \quad \text{open cover of } X.$$

X compact, there is a finite subcover:

$$\exists p_1, p_2, p_3, \dots p_\ell \text{ s.t. } X \subseteq \bigcup_{i=1}^{\ell} V_{p_i}$$

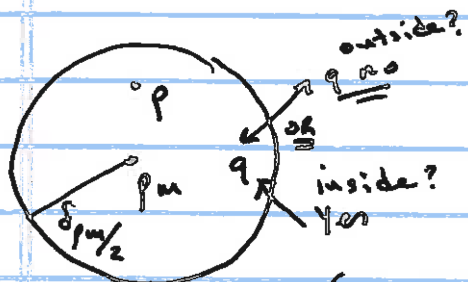
Let $\delta = \frac{1}{2} \min(\delta_{p_1}, \delta_{p_2}, \dots \delta_{p_\ell}) > 0$

Let $p, q \in X$ s.t. $d_X(p,q) < \delta$

$\left(\underline{\text{WTS}}: \ d_Y(f(p), f(q)) < \varepsilon \right)$

$p \in X \subseteq \bigcup_{i=1}^{l} V_{p_i}$, so $\exists\, m$ $1 \le m \le l$ s.t. $p \in V_{p_m}$

$$d(p, p_m) < \frac{\delta_{p_m}}{2}$$



$V_{p_m} = N_{\frac{\delta_{p_m}}{2}}(p_m)$

$$\begin{aligned} d(q, p_m) &\le d(p,q) + d(p, p_m) \\ &< \delta + \frac{\delta_{p_m}}{2} \\ &\le \frac{\delta_{p_m}}{2} + \frac{\delta_{p_m}}{2} = \delta_{p_m} \end{aligned}$$

$\Longrightarrow$ $d(q, p_m) < \delta_{p_m}$ so (*) holds.

$$d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q))$$

by (*). $\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ #

Thm: 4.20 HW to read (p91–92)

Different functions from the text book

① E is not closed $\exists p \in E' - E$

(a) $f : E \to \mathbb{R}$ $\quad f(x) = \dfrac{1}{|x-p|}$ unbounded since $\exists p_n \in E$
$p_n \to p \notin E$

(b) $g : E \to \mathbb{R}$ $\quad g(x) = \dfrac{1}{1+|x-p|} < 1 \quad p \notin E$

$\sup g = 1.$ since $g(p_n) \to 1.$

② If E is unbounded, take

$f(x) = |x|$ for (a)

$g(x) = \dfrac{|x|}{1+|x|}$ for (b)

Ex Let (X, d) be any metric space, p_0 be any pt.

$p \quad \cdot q$

$f : X \to [0,\infty)$ $\quad f(q) = d(q, p_0) : X \to [0,\infty)$
is uniformly continuous

p_0 $\quad \forall \varepsilon > 0 \ \exists \delta = \varepsilon$

reverse Δ-ineq.

$$d_X(p,q) < \delta \implies |f(p) - f(q)| = |d(p_0,p) - d(p_0,q)| \le d(p,q) < \delta = \varepsilon.$$

CONTINUITY & CONNECTEDNESS

4.22 **Thm:** Let $f:(\mathbb{X},d_{\mathbb{X}})\to(Y,d_Y)$ be continuous.

$\left.\begin{array}{l} E \text{ is connected} \\ E\subseteq\mathbb{X}\end{array}\right\} \Longrightarrow f(E)$ is connected.

Recall 1) $A,B\subseteq\mathbb{X}$, A & B called seperated if $\bar{A}\cap B=\emptyset=A\cap\bar{B}$

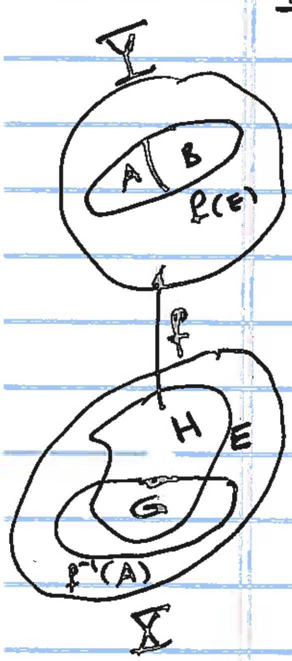
2) A set F is called connected if $\nexists$ no A,B s.t. i) $A\neq\emptyset$, $B\neq\emptyset$

ii) $F=A\cup B$

iii) A & B are separated.

Proof: Suppose $\exists$ a separation A,B of $f(E)$

$\left.\begin{array}{l} \cdot\ f(E)=A\cup B \\ \cdot\ \bar{A}\cap B=\bar{B}\cap A=\emptyset \\ \cdot\ A\neq\emptyset,\ B\neq\emptyset \end{array}\right\}$ (WTS: $\exists$ a separation of E)



Let $G=f^{-1}(A)\cap E$

$H=f^{-1}(B)\cap E$

$A\subseteq\bar{A}$

$G=f^{-1}(A)\cap E\subseteq f^{-1}(A)\subseteq \underbrace{f^{-1}(\bar{A})}_{\text{closed}}$ $\Leftarrow$ $\bar{A}$ closed, f cont.

$G\subseteq f^{-1}(\bar{A})$

$\bar{G}\subseteq f^{-1}(\bar{A})$

$$H \subseteq f^{-1}(B)$$

$$\bar{G} \cap H \subseteq f^{-1}(\bar{A}) \cap f^{-1}(B) = f^{-1}(\underbrace{\bar{A} \cap B}_{\phi}) = \phi.$$

(1) $\left[\begin{array}{l} \bar{G} \cap H = \phi. \\ \bar{H} \cap G = \phi \quad \text{similarly} \end{array}\right.$

(2) $\left[\begin{array}{l} G \cup H = (f^{-1}(A) \cap E) \cup (f^{-1}(B) \cap E) \\ \quad = \big(\underbrace{f^{-1}(A) \cup f^{-1}(B)}_{\text{contains } E \text{ since } A \cup B = f(E)}\big) \cap E = E. \end{array}\right.$

(3) $\left[\begin{array}{l} A \neq \phi \ \& \ A \subseteq f(E) \implies f^{-1}(A) \neq \phi \\ \quad \exists a \in A \neq \phi, \ a = f(c) \text{ for some } c \in E \\ \quad \exists c \in f^{-1}(A) \cap E = G \neq \phi. \end{array}\right.$

Similarly $\quad H \neq \phi.$

(1), (2), (3) $\Rightarrow$ G, H is a separation of E.

E is connected. So such G & H must not exist.

Hence $f(E)$ is connected.

(Since A separation of $f(E)$ with A & B implies a separation of E with G & H.)