March 23, 2018

HW questions:

p 80 #13 Chap III

$\sum a_n$ absolutely convergent $\Rightarrow \sum |a_n|$ is convergent

$\sum b_n$ " " $\Rightarrow \underbrace{\sum |b_n| \text{ is convergent}}_{*}$

$$\sum_{n=0}^{\infty} c_n, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

$$|c_n| = \left|\sum_{k=0}^{n} a_k b_{n-k}\right| \leq \sum_{k=0}^{n} |a_k||b_{n-k}| = d_n.$$

$\sum_{n=0}^{\infty} d_n$ converges by 3.50 ← by using *

$\sum_{n=0}^{\infty} |c_n|$ converges by comparison test.

$\sum_{n=0}^{\infty} c_n$ " absolutely

Chap III #8

Case 1:

$b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \longrightarrow b.$

$b_n' = b_n - b \qquad b_n' \downarrow 0$

$\sum a_n$ convergent $\Rightarrow A_n = \sum_{k=1}^{n} a_k$ convergent

$\Rightarrow \{A_n\}$ bounded.

Thm 3.42

$$\underbrace{\sum_{n=1}^{\infty} b_n' a_n}_{\text{convergent}} + b \underbrace{\sum_{n=1}^{\infty} a_n}_{\text{convergent}} \Rightarrow \underbrace{= \sum_{n=1}^{\infty} (b_n - \cancel{b} + \cancel{b}) a_n}_{\text{convergent}}.$$

Case 2 $\quad b_1 \leq b_2 \leq \cdots \leq b_n \leq b_{n+1} \cdots \rightarrow b$

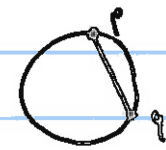
multiply with (-1) to reduce to the first case.

Ex

$$f : \underbrace{[0, 2\pi)}_{X} \longrightarrow S^1 = \{(x,y) \mid x^2 + y^2 = 1\}$$

$$[0,2\pi) \subseteq (\mathbb{R}, \|\cdot\|)$$

$p, q \in S^1 \qquad d(p,q) = |p - q|$



$f(t) = (\cos t, \sin t)$

f continuous on $[0, 2\pi)$

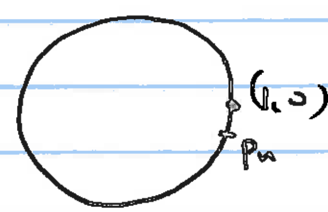
f is 1-1

f is onto.

$\exists$ inverse $g : S^1 \longrightarrow X$

$(\cos t, \sin t) \longmapsto t$.

<u>g is not continuous:</u> $\quad p_n = f\left(2\pi - \frac{1}{n}\right)$



$p_n \longrightarrow (1,0)$

$2\pi - \frac{1}{n} = g(p_n) \not\longrightarrow g(1,0) = 0$

X is not compact

S^1 is compact.

See next page/ Thm, compare.

(4.17) **Thm:**

Let $f : (X, d_X) \to (Y, d_Y)$
be 1-1, onto, and continuous.
Let $g : (Y, d_Y) \to (X, d_X)$ be
defined by $g(y) = x \iff f(x) = y$

If X is compact then g is continuous.

Defn A bijective map $f : (X, d_X) \to (Y, d_Y)$ is called a homeomorphism if both f and f^{-1} are continuous.

Proof: $g : (Y, d_Y) \to (X, d_X)$
(WTS) $\forall$ closed set $A \subseteq X$, $g^{-1}(A)$ is closed in Y.
(which would imply g is continuous.)

Let A be a closed set in X, which is compact,
A is compact.
$f(A)$ is compact since f is continuous.
$f(A)$ is closed.

$Y \supseteq f(A) = \{ f(x) \mid x \in A \}$ ← same set since
$Y \supseteq g^{-1}(A) = \{ y \mid g(y) \in A \}$ ←
$g(y) = x \iff f(x) = y$

$g^{-1}(A)$ closed in Y.

Hence g is continuous by Corollary of Thm 4.8, p 87.

Defn Let $f:(X,d_X)\to (Y,d_Y)$.

f is called uniformly continuous on X, if

Choice of δ does NOT depend on x. → $\forall \varepsilon>0\ \exists \delta>0\ \forall x,y\in X,\ d_X(x,y)<\delta \Rightarrow d_Y(f(x),f(y))<\varepsilon$

→ Compare to continuous

$\forall x\in X\ \forall \varepsilon>0\ \exists\delta>0\ \forall y\in X\ d_X(x,y)<\delta \Rightarrow d_Y(f(x),f(y))<\varepsilon$

Choice of δ may depend on x.

Obs Unif cont $\Rightarrow$ continuous

Continuous $\not\Rightarrow$ unif continuous, in general.

Thm: $\left.\begin{array}{l}\text{Continuous}\\ X \text{ compact}\end{array}\right\}\Rightarrow$ unif continuous

Ex $f(x)=\frac{1}{x}\ ;\ (0,\infty)\to(0,\infty)$ is not unif. cont.

Want not $(\forall\varepsilon>0\ \exists\delta>0\ \forall x,y\in X\ ;\ d_X(x,y)<\delta \Rightarrow d_Y(f(x),f(y))<\varepsilon)$

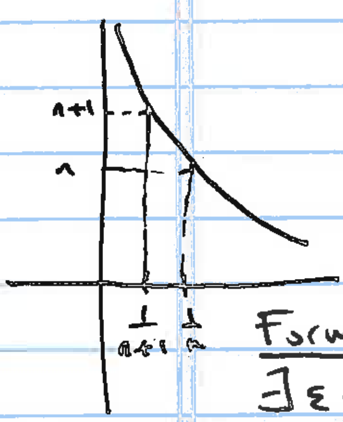
Want $\exists\varepsilon>0\ \forall\delta>0\ \exists x,y\in X\quad d_X(x,y)<\delta$ and $d_Y(f(x),f(y))\geq\varepsilon$

So: $\exists\ \varepsilon=1\quad \forall\delta\approx\frac{1}{n^2}\quad \exists x_n=\frac{1}{n},\ y_n=\frac{1}{n+1}$

$|x_n-y_n| = \frac{1}{n}-\frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}\leq\delta$

$|f(x_n)-f(y_n)| = |n-(n+1)| = 1$

} prep.



Formal proof:

$\exists\varepsilon=1>0\ \forall\delta>0\ \exists n\in\mathbb{N}\ \frac{1}{n^2}\leq\delta\quad \exists\ x_n=\frac{1}{n},\ y_n=\frac{1}{n+1}$ s.t.

$|x_n-y_n|=\frac{1}{n(n+1)}<\frac{1}{n^2}\leq\delta$ and $|f(x_n)-f(y_n)|=1\geq\varepsilon$

Fixed $A, B, C \in \mathbb{R}$.

Ex $f(x,y) = Ax + By + C : \mathbb{R}^2 \to \mathbb{R}^1$ is uniformly continuous.

$\forall \varepsilon > 0 \ \exists \delta = \frac{\varepsilon}{|A| + |B| + 1} > 0$

$\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$|(x_1, y_1) - (x_2, y_2)| < \delta \implies |x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$

$|f(x_1, y_1) - f(x_2, y_2)| = |(Ax_1 + By_1 + C) - (Ax_2 + By_2 + C)|$

$= |A(x_1 - x_2) + B(y_1 - y_2)|$

$\leq |A||x_1 - x_2| + |B||y_1 - y_2|$

$< (|A| + |B|)\delta = (|A| + |B|) \cdot \frac{\varepsilon}{|A| + |B| + 1} \leq \varepsilon$